Nonlinear Analysis 72 (2010) 1110–1119

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

The dynamic programming approach to multi-model robust optimization

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ARTICLE INFO

Article history:
Received 20 November 2008
Accepted 16 July 2009

Keywords:
Multi-model control systems
Optimal control
The dynamic programming

ABSTRACT

The aim of this paper is to extend the dynamic programming (DP) approach to multi-model optimal control problems (OCPs). We deal with robust optimization of multi-model control systems and are particularly interested in the Hamilton–Jacobi–Bellman (HJB) equation for the above class of problems. In this paper, we study a variant of the HJB for multi-model OCPs and examine the natural relationship between the Bellman DP techniques and the Robust Maximum Principle (MP). Moreover, we describe how to carry out the practical calculations in the context of multi-model LQ-problems and derive the associated Riccati-type equation.

1. Introduction

The theory of OCPs governed by ordinary differential equations has been well established since the middle of the 20th century; see e.g., [1–6] and the references therein. For a classical OCP, the main tools toward the construction of optimal trajectories, and then optimal synthesis, are the celebrated Pontryagin MP and the Bellman DP. Recently robust optimization problems for multi-model control systems have attracted a lot of attention; thus both theoretical results and applications were developed, (see [7–11]). OCPs for multi-model dynamical systems arise in the control of mechanical multibody systems, electrical circuits and heterogeneous systems, where different models are coupled together. The majority of applied OCPs are problems with incomplete information on the model structure or parameters. The multi-model control systems provide useful theoretical models for some classes of dynamical systems with the above-mentioned types of uncertainties. In this case one of the most efficient approaches to the optimal design of such systems is the robust optimization technique. Optimal robust control strategies based on the minimax algorithms have found a wide use in the design of complex control systems. Robust MP proposed by Boltyanski and Poznyak (see e.g., [7–11]) is the basic analytical result for studying OCPs with multi-model controlled plants. On the other hand, the Bellman DP techniques are not far enough advanced for multi-model OCPs.

The purpose of this paper is to apply the classic DP techniques to a class of multi-model OCPs. First, we verify the Bellman *principle of optimality* for the class of problems under consideration. Second, we derive a (robust) version of the HJB equation. It should be stressed that our main result deals with a finite parametric set involved into a model description. We also apply the HJB equation to a multi-model LQ-problem (see [9,10]) and derive a parametric Riccati equation for the linear-quadratic case. Moreover, the obtained theoretical facts are considered in comparison with the corresponding theorem resulting from the application of the Robust MP to multi-model LQ-problems [10]. In such a manner we establish the natural relationship between DP and the Robust MP for the given class of LQ-problems (see e.g., [12]).

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doi:10.1016/j.na.2009.07.050

The remainder of our paper is organized as follows. Section 2 contains a problem formulation, some basic concepts and preliminary results. Section 3 is devoted to the main result of this paper, namely, to a variant of the HJB equation for multi-model OCPs. Moreover, we also deal with the corresponding verification techniques. In Section 4 we apply our theoretical results to the multi-model linear quadratic problems and deduce a Riccati-formalism similar to the classic LQ-theory. Section 5 summarizes the paper.

2. Problem formulation and preliminary results

Consider the following initial-value problem for a multi-model control system

$$\begin{aligned} \dot{x}(t) &= f^{\alpha}(t, x(t), u(t)) \quad \text{a.e. on } [0, t_f], \\ x(0) &= x_0, \end{aligned} \tag{1}$$

where $f^{\alpha} : [0, t_f] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ for every α from a finite parametric set $\mathcal{A}$, $u(t) \in U$ and $x_0 \in \mathbb{R}^n$ is a fixed initial state. Note that parameter α indicates the corresponding "model" (or "realization") of the multi-model system under consideration (see [7–10]). We assume that U is a compact subset of $\mathbb{R}^m$ and introduce the set of admissible control functions

$$\mathcal{U} := \{u(\cdot) \in \mathbb{L}^{\infty}_m([0, t_f]) \;:\; u(t) \in U \quad \text{a.e. on } [0, t_f]\}.$$

Here $\mathbb{L}^{\infty}_m([0, t_f])$ is the standard Lebesgue space of (bounded) measurable control functions $u : [0, t_f] \to \mathbb{R}^m$ such that $\operatorname{ess\,sup}_{t \in [0, t_f]} \|u(t)\|_{\mathbb{R}^m} < \infty$. In addition, we assume that for each $\alpha \in \mathcal{A}$, $u(\cdot) \in \mathcal{U}$ the realized initial-value problem (1) has a unique absolutely continuous solution $x^{\alpha,u}(\cdot)$. For some constructive existence and uniqueness conditions see e.g., [4, 6]. Let $u(\cdot)$ be an admissible control function. This control gives rise to the complete dynamic of the given multi-model system (1), and we can define the $(n \times |\mathcal{A}|)$-dimensional "state vector" of system (1)

$$X^u(t) := \left(x^{\alpha_1,u}(t), \ldots, x^{\alpha_{|\mathcal{A}|},u}(t)\right)_{\alpha \in \mathcal{A}}, \quad t \in [0, t_f].$$

In a similar way we consider a "trajectory" of system (1) as an absolutely continuous $(n \times |\mathcal{A}|)$-dimensional function $X^u(\cdot)$. In the following, we also will use the following notation

$$\begin{aligned} F(t, X, u) &:= \left(f^{\alpha_1}(t, x, u), \ldots, f^{\alpha_{|\mathcal{A}|}}(t, x, u)\right)_{\alpha \in \mathcal{A}}, \\ h(u(\cdot), x^{\alpha,u}(\cdot)) &:= \int_0^{t_f} f_0(t, x^{\alpha,u}(t), u(t))\mathrm{d}t, \end{aligned}$$

where $f_0 : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is a continuous function (the integrand of the cost functional). Clearly, functional $h(u(\cdot), x^{\alpha,u}(\cdot))$ is associated with the corresponding realized model from (1). If we assume that the realized value of the parameter α is unknown, then the *worst cost* (highest cost) can be easily defined as

$$J(u(\cdot)) := \max_{\alpha \in \mathcal{A}} h(u(\cdot), x^{\alpha,u}(\cdot)).$$

Note that the "common" cost functional J depends only on the given admissible control $u(\cdot)$. Let us now formulate the robust (minimax) OCP for a multi-model control system

$$\begin{aligned} &\text{minimize } J(u(\cdot)) \\ &\text{subject to (1)}, \quad \alpha \in \mathcal{A}, u(\cdot) \in \mathcal{U}. \end{aligned} \tag{2}$$

A pair $(u(\cdot), X^u(\cdot))$, where $u(\cdot) \in \mathcal{U}$, is called an *admissible process* for (2). Note that we consider admissible processes defined on the (finite) time-interval $[0, t_f]$.

Remark 1. Roughly speaking, in the context of problem (2) we are interested in a control strategy which provides a "good" behavior for all systems from the given collection of models (1) even in the "worst" cost case. The resulting control strategy is applied to every α-model from (1) simultaneously. A solution of (2) guarantees an optimal robust behavior of the corresponding multi-model system in the sense of the above cost functional J. Note that one can theoretically consider a control design determined by the following optimization procedure: "maximize $\min_{u(\cdot) \in \mathcal{U}} h(u(\cdot), x^{\alpha,u}(\cdot))$ subject to (1), $\alpha \in \mathcal{A}$, $u(\cdot) \in \mathcal{U}$". Evidently, a solution to this last (maximin) OCP cannot be interpreted as a robust optimization in the framework of the above-mentioned "worst case". Moreover, a control generated by this maximin optimization procedure possesses (in general) the optimality property only for some models from (1). Therefore, a simultaneously application of the above maximin-control to all α-models from (1) (for all $\alpha \in \mathcal{A}$) does not lead to an adequate optimal dynamics for all systems from the collection (1).

Multi-model OCPs of the Bolza-type have been studied in [10]. Let us examine the Bolza cost functional associated with system (1)

$$\tilde{h}(u(\cdot), x^{\alpha,u}(\cdot)) := \phi(x^{\alpha,u}(t_f)) + \int_0^{t_f} \tilde{f}_0(t, x^{\alpha,u}(t), u(t))\mathrm{d}t,$$

where $\phi : \mathbb{R}^n \to \mathbb{R}$ is a continuously differentiable function (a smooth terminal term) and $\tilde{f}_0$ is a continuous function. Note that we deal here with a classical Bolza functional $\tilde{h}$. The smooth function ϕ introduced above characterizes the possible