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Continuum shape sensitivity analysis of mixed-mode fracture using fractal finite element method

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Abstract

This paper presents a new fractal finite element based method for continuum-based shape sensitivity analysis for a crack in a homogeneous, isotropic, and two-dimensional linear-elastic body subject to mixed-mode (modes I and II) loading conditions. The method is based on the material derivative concept of continuum mechanics, and direct differentiation. Unlike virtual crack extension techniques, no mesh perturbation is needed in the proposed method to calculate the sensitivity of stress-intensity factors. Since the governing variational equation is differentiated prior to the process of discretization, the resulting sensitivity equations predicts the first-order sensitivity of *J*-integral or mode-I and mode-II stress-intensity factors, K_{I} and K_{II}, more efficiently and accurately than the finite-difference methods. Unlike the integral based methods such as *J*-integral or *M*-integral no special finite elements and post-processing are needed to determine the first-order sensitivity of *J*-integral or K_{I} and K_{II}. Also a parametric study is carried out to examine the effects of the similarity ratio, the number of transformation terms, and the integration order on the quality of the numerical solutions. Four numerical examples which include both mode-I and mixed-mode problems, are presented to calculate the first-order derivative of the *J*-integral or stress-intensity factors. The results show that first-order sensitivities of *J*-integral or stress-intensity factors obtained using the proposed method are in excellent agreement with the reference solutions obtained using the finite-difference method for the structural and crack geometries considered in this study.

Keywords: Crack; Fractal finite element method; Stress-intensity factor; Linear-elastic fracture mechanics; Mixed-mode; Shape sensitivity analysis; Velocity field; Material derivative

1. Introduction

In recent years methods based on fractal geometry concepts to generate infinite number of finite elements around the crack tip to capture the crack-tip singularity have been developed or investigated to solve linear-elastic fracture-mechanics (LEFM) problems [1–5]. The fractal finite element method (FFEM) is one such method developed for calculating the stress-intensity factors (SIFs) in linear-elastic crack problems. In its original form, the fractal two-level finite element method was first proposed by Leung and Su in 1993 [6]. Since

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Nomenclature

a	crack length
$\boldsymbol{a}$	William's eigenfunction series coefficients $\{a_0^{\mathrm{I}}, a_0^{\mathrm{II}}, a_1^{\mathrm{I}}, a_1^{\mathrm{II}}, a_2^{\mathrm{I}}, a_2^{\mathrm{II}}, \ldots\}^{\mathrm{T}}$
$\dot{\boldsymbol{a}}$	material derivative of $\boldsymbol{a}$
$a'_V(\cdot,\cdot)$	structural fictitious load form
$a_\Omega(\cdot,\cdot)$	structural energy form
$\boldsymbol{B}$	strain displacement matrix
$\boldsymbol{d}$	nodal displacement vector
$\boldsymbol{d}_m$	displacement vector of the master nodes
$\boldsymbol{d}_r$	displacement vector of the nodes in the regular region other than the master nodes
$\boldsymbol{d}_s^{k\mathrm{th}}$	nodal displacement vector for the second and subsequent layers in the singular region
$\boldsymbol{d}_s^{\prime k\mathrm{th}}$	nodal displacement sensitivity vector for the second and subsequent layers in the singular region
$\boldsymbol{d}_s^{1\mathrm{st}}$	displacement vector of the slave nodes in the first layer of the singular region
$\dot{\boldsymbol{d}}$	material derivative of nodal displacement vector
$\dot{\boldsymbol{d}}_m$	material derivative of displacements of the master nodes
$\dot{\boldsymbol{d}}_r$	material derivative of displacements of the nodes in the regular region other than the master nodes
$\dot{\boldsymbol{d}}_s^{1\mathrm{st}}$	material derivative of displacements of the slave nodes in the first layer of the singular region
$\boldsymbol{D}$	constitutive tensor
E	Young's modulus
$\boldsymbol{f}$	nodal force vector
$\boldsymbol{f}_s^{k\mathrm{th}}$	nodal force vector for the second and subsequent layers in the singular region
$\boldsymbol{f}_r^R, \boldsymbol{f}_m^R$	partitioned force vector in the regular region with respect to the nodes other than the master nodes and the master nodes
$\boldsymbol{f}_m^{1\mathrm{st}}, \boldsymbol{f}_s^{1\mathrm{st}}$	partitioned force vector for the first layer in the singular region with respect to the master nodes and the slave nodes
$\boldsymbol{f}'$	global force sensitivity vector
$\boldsymbol{f}_s^{\prime k\mathrm{th}}$	nodal force sensitivity vector for the second and subsequent layers in the singular region
$\boldsymbol{f}_r^{\prime R}, \boldsymbol{f}_m^{\prime R}$	partitioned force sensitivity vectors in the regular region with respect to the nodes other than the master nodes and the master nodes
$\boldsymbol{f}_m^{\prime 1\mathrm{st}}, \boldsymbol{f}_s^{\prime 1\mathrm{st}}$	partitioned force sensitivity vectors for the first layer in the singular region with respect to the master nodes and the slave nodes
$\boldsymbol{f}'_I$	element-level force sensitivity vector
$G_{ij}(n,\theta)$	angular functions
K_{I}	mode-I stress-intensity factor
K_{II}	mode-II stress-intensity factor
$\partial K_{\mathrm{I}}/\partial a$	sensitivity of Mode-I stress-intensity factor
$\partial K_{\mathrm{II}}/\partial a$	sensitivity of Mode-II stress-intensity factor
$\boldsymbol{k}'_{IJ}$	element-level stiffness sensitivity matrix
$\boldsymbol{K}$	stiffness matrix
$\boldsymbol{K}_s^{k\mathrm{th}}$	stiffness matrix for the second and subsequent layers in the singular region
$\boldsymbol{K}_{rr}^R, \boldsymbol{K}_{rm}^R, \boldsymbol{K}_{mr}^R, \boldsymbol{K}_{mm}^R$	partitioned stiffness matrices in the regular region with respect to the nodes other than the master nodes and the master nodes
$\boldsymbol{K}_{mm}^{1\mathrm{st}}, \boldsymbol{K}_{ms}^{1\mathrm{st}}, \boldsymbol{K}_{sm}^{1\mathrm{st}}, \boldsymbol{K}_{ss}^{1\mathrm{st}}$	partitioned stiffness matrices for the first layer in the singular region with respect to the master nodes and the slave nodes
$\boldsymbol{K}'$	global stiffness sensitivity matrix
$\boldsymbol{K}_s^{\prime k\mathrm{th}}$	stiffness sensitivity matrix for the second and subsequent layers in the singular region