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Topological sensitivity analysis of inclusion in two-dimensional linear elasticity

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ABSTRACT

The topological derivative gives the sensitivity of the problem when the domain under consideration is perturbed by the introduction of a hole. Alternatively, this same concept can also be used to calculate the sensitivity of the problem when, instead of a hole, a small inclusion is introduced at a point in the domain. In the present paper we apply the Topological-Shape Sensitivity Method to obtain the topological derivative of inclusion in two-dimensional linear elasticity, adopting the total potential energy as the cost function and the equilibrium equation as a constraint. For the sake of completeness, initially we present a brief description of the Topological-Shape Sensitivity Method. Then, we calculate the topological derivative for the problem under consideration in two steps: firstly we perform the shape derivative and next we calculate the limit when the perturbation vanishes using classical asymptotic analysis around a circular inclusion. In addition, we use this information as a descent direction in a topology design algorithm which allows to simultaneously remove and insert material. Finally, we explore this feature showing some numerical experiments of structural topology design within the context of two-dimensional linear elasticity problem.

1. Introduction

As it is understood, the topological derivative furnishes the sensitivity of the problem when the domain under consideration is perturbed by the introduction of a hole [1–4]. This methodology has been recognized as an alternative and at the same time a promising tool to solve topology optimization problems (see, for instance, [5] and references therein). Moreover, this is a broad concept. In fact, the topological derivative may also be applied to analyze any kind of sensitivity problem in which, instead of a hole, discontinuous changes in a small region are allowable; for example, discontinuous changes on the shape of the boundary, on the boundary conditions, on the load system and/or on the parameters of the problem. In particular when the parameter is related to material property, we can calculate the topological derivative of inclusion [6], instead of a hole.

Therefore, the information provided by the topological derivative is also very effective to solve problems such as image processing (enhancement and segmentation) [7–10], inverse problems (domain, boundary conditions and parameters characterization) [11–16] and in the mechanical modeling of problems with changes on the configuration of the domain such as fracture mechanics and damage.

Several methods were proposed to calculate the topological derivative [1,4,6,17]. In the present work we extend the application of the Topological-Shape Sensitivity Method developed in [6] to obtain the topological derivative of inclusion in two-dimensional linear elasticity, adopting the total potential energy as the cost function and the equilibrium equation as the constraint. Next, we apply this result to devise a topology design algorithm which allows us to simultaneously remove and insert material. This feature is demonstrated through several numerical experiments.

This study is organized in the following manner. In Section 2, we present a brief description of the Topological-Shape Sensitivity Method. In Section 3, we calculate the topological derivative of inclusion for the problem under consideration. Lastly, in Section 4, we show some numerical results concerning structural topology design.

2. Topological-shape sensitivity method

Let us consider an open bounded domain $\Omega \subset \mathbb{R}^2$ with a smooth boundary $\partial\Omega$. If the domain Ω is perturbed by introducing a small inclusion represented by B_ε, which is a ball of radius ε centered at point $\hat{\mathbf{x}} \in \Omega$, we have a perturbed domain $\Omega_\varepsilon \cup B_\varepsilon$,

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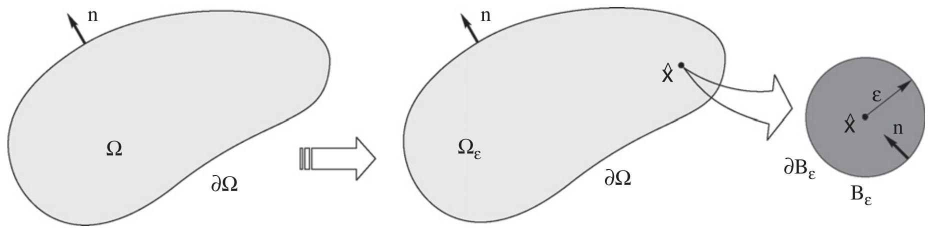

Fig. 1. Topological derivative concept.

where $\Omega_\varepsilon = \Omega - B_\varepsilon$ and $\partial\Omega_\varepsilon = \partial\Omega \cup \partial B_\varepsilon$, as shown in Fig. 1. Thus, considering a cost function ψ defined in both domains Ω and $\Omega_\varepsilon \cup B_\varepsilon$, its topological derivative is written as

$$D_T(\hat{\mathbf{x}}) = \lim_{\varepsilon\to 0} \frac{\psi(\Omega_\varepsilon \cup B_\varepsilon) - \psi(\Omega)}{f(\varepsilon)}, \tag{1}$$

where $f(\varepsilon)$ is a function that decreases monotonically so that $f(\varepsilon) \to 0$ with $\varepsilon \to 0^+$.

Recently, an alternative procedure to calculate the topological derivative, called Topological-Shape Sensitivity Method, has been introduced by the authors (see, for instance, [3,18]). This approach makes use of the whole mathematical framework (and results) developed for shape sensitivity analysis (see, for instance, the pioneering work of Murat and Simon [19]). Main results obtained in [3,18] are briefly summarized in the following Theorem (see also [6]):

Theorem 1. *Let $f(\varepsilon)$ be a function chosen in order to $0<|D_T(\hat{\mathbf{x}})|<\infty$, then the topological derivative given by Eq. (1) can be written as*

$$D_T(\hat{\mathbf{x}}) = \lim_{\varepsilon\to 0} \frac{1}{f'(\varepsilon)} \frac{\mathrm{d}}{\mathrm{d}\tau}\, \psi(\Omega_\tau)\Big|_{\tau=0}, \tag{2}$$

where $\tau \in \mathbb{R}^+$ is used to parameterize the domain. That is, for τ small enough, we have

$$\Omega_\tau := \{\mathbf{x}_\tau \in \mathbb{R}^2 : \mathbf{x}_\tau = \mathbf{x} + \tau\mathbf{v},\ \mathbf{x} \in \Omega_\varepsilon \cup B_\varepsilon\}. \tag{3}$$

Therefore, $\mathbf{x}_\tau|_{\tau=0} = \mathbf{x}$ and $\Omega_\tau|_{\tau=0} = \Omega_\varepsilon \cup B_\varepsilon$. In addition, considering that $\mathbf{n}$ is the outward normal unit vector (see Fig. 1), then we can define the shape change velocity $\mathbf{v}$, *which is a smooth vector field in $\Omega_\varepsilon \cup B_\varepsilon$ assuming the following values on the boundary ∂B_ε and $\partial\Omega$*

$$\begin{cases} \mathbf{v} = -\mathbf{n} & \text{on } \partial B_\varepsilon, \\ \mathbf{v} = \mathbf{0} & \text{on } \partial\Omega, \end{cases} \tag{4}$$

and the shape sensitivity of the cost function in relation to the domain perturbation characterized by $\mathbf{v}$ is given by

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\psi(\Omega_\tau)\Big|_{\tau=0} = \lim_{\tau\to 0} \frac{\psi(\Omega_\tau) - \psi(\Omega_\varepsilon \cup B_\varepsilon)}{\tau}. \tag{5}$$

Proof. The reader interested in the proof of this result may refer to [6]. □

3. The topological derivative of inclusion

To highlight the capabilities of the Topological-Shape Sensitivity Method, it will be applied to calculate the topological derivative of inclusion in two-dimensional linear elasticity considering the total potential energy as the cost function and the equilibrium equation in its weak form as the constraint. Therefore, considering the above problem, initially we perform the shape sensitivity of the adopted cost function with respect to a change in the shape of the inclusion and finally we calculate the associated topological derivative.

3.1. Shape sensitivity analysis

Let us choose the total potential energy stored in the elastic solid under analysis as the cost function. For simplicity, we assume that the external load remains fixed during the shape change. As it is well known, different approaches can be applied to obtain the shape derivative of the cost function. However, in our particular case, as the cost function is associated with the potential of the state equation, the direct differentiation method will be adopted to calculate its shape derivative. Therefore, considering the total potential energy already written in the configuration Ω_τ, Eq. (3), then $\psi(\Omega_\tau) := \mathcal{J}_{\Omega_\tau}(\mathbf{u}_\tau)$: $\mathcal{U}_\tau \mapsto \mathbb{R}$ can be expressed by

$$\mathcal{J}_{\Omega_\tau}(\mathbf{u}_\tau) = \frac{1}{2}\int_{\Omega_\tau} \mathbf{T}_\tau(\mathbf{u}_\tau)\cdot\mathbf{E}_\tau(\mathbf{u}_\tau)\,\mathrm{d}\Omega_\tau - \int_{\Gamma_N} \bar{\mathbf{q}}\cdot\mathbf{u}_\tau\,\mathrm{d}\Gamma_\tau, \tag{6}$$

where the admissible displacements set $\mathcal{U}_\tau$ is given by

$$\mathcal{U}_\tau = \{\mathbf{u}_\tau \in [H^1(\Omega_\tau)]^2 : \mathbf{u}_\tau = \bar{\mathbf{u}} \text{ on } \Gamma_D\}. \tag{7}$$

The strain and stress tensors $\mathbf{E}_\tau(\mathbf{u}_\tau)$ and $\mathbf{T}_\tau(\mathbf{u}_\tau)$ are, respectively, given by

$$\mathbf{E}_\tau(\mathbf{u}_\tau) = \nabla^s_\tau \mathbf{u}_\tau \quad\text{and}\quad \mathbf{T}_\tau(\mathbf{u}_\tau) = \mathbf{C}_\delta \nabla^s_\tau \mathbf{u}_\tau, \tag{8}$$

with $\nabla_\tau(\cdot)$ used to denote

$$\nabla_\tau(\cdot) := \frac{\partial}{\partial \mathbf{x}_\tau}(\cdot), \tag{9}$$

and the elasticity tensor $\mathbf{C}_\delta$ is defined as follows:

$$\mathbf{C}_\delta = \frac{K_\delta}{1-\nu^2}\,[(1-\nu)\mathbf{II} + \nu(\mathbf{I}\otimes\mathbf{I})], \tag{10}$$

where $\mathbf{I}$ and $\mathbf{II}$ are, respectively, the second and fourth order identity tensors, ν is Poisson's ratio and, for $\delta \in \mathbb{R}^+$, K_δ is Young's modulus given by

$$K_\delta = \begin{cases} K & \text{if } \mathbf{x} \in \Omega_\varepsilon, \\ \delta K & \text{if } \mathbf{x} \in B_\varepsilon. \end{cases} \tag{11}$$

In addition, $\mathbf{u}_\tau$ is the solution of the variational problem defined in the configuration Ω_τ, that is: find the displacement vector field $\mathbf{u}_\tau \in \mathcal{U}_\tau$ such that

$$\int_{\Omega_\tau} \mathbf{T}_\tau(\mathbf{u}_\tau)\cdot\mathbf{E}_\tau(\boldsymbol{\eta}_\tau)\,\mathrm{d}\Omega_\tau = \int_{\Gamma_N} \bar{\mathbf{q}}\cdot\boldsymbol{\eta}_\tau\,\mathrm{d}\Gamma_\tau \quad \forall \boldsymbol{\eta}_\tau \in \mathcal{V}_\tau, \tag{12}$$

where

$$\mathcal{V}_\tau = \{\boldsymbol{\eta}_\tau \in [H^1(\Omega_\tau)]^2 : \boldsymbol{\eta}_\tau = \mathbf{0} \text{ on } \Gamma_D\}. \tag{13}$$

Observe that from the well-known terminology of Continuum Mechanics, the domains $\Omega_\tau|_{\tau=0} = \Omega_\varepsilon \cup B_\varepsilon$ and Ω_τ can be interpreted as the material and the spatial configurations, respectively. Therefore, in order to calculate the shape derivative of the cost function $\mathcal{J}_{\Omega_\tau}(\mathbf{u}_\tau)$, at $\tau = 0$, we may use Reynolds' transport theorem and the concept of material derivatives of spatial fields,