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The derivative based variance sensitivity analysis for the distribution parameters and its computation

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A B S T R A C T

The output variance is an important measure for the performance of a structural system, and it is always influenced by the distribution parameters of inputs. In order to identify the influential distribution parameters and make it clear that how those distribution parameters influence the output variance, this work presents the derivative based variance sensitivity decomposition according to Sobol′s variance decomposition, and proposes the derivative based main and total sensitivity indices. By transforming the derivatives of various orders variance contributions into the form of expectation via kernel function, the proposed main and total sensitivity indices can be seen as the "by-product" of Sobol′s variance based sensitivity analysis without any additional output evaluation. Since Sobol′s variance based sensitivity indices have been computed efficiently by the sparse grid integration method, this work also employs the sparse grid integration method to compute the derivative based main and total sensitivity indices. Several examples are used to demonstrate the rationality of the proposed sensitivity indices and the accuracy of the applied method.

1. Introduction

Development of probabilistic sensitivity analysis is frequently considered as an essential component of a probabilistic analysis, and it is often critical towards understanding the physical mechanisms and modifying the design to mitigate and manage risk [1]. Traditional sensitivity analysis (SA) can be classified into two groups: local SA and global SA [2].

Local SA usually investigates how small variation of a distribution parameter around a reference point changes the value of the output. One classical local SA is the derivative based SA by defining the derivative of probabilistic statistics quantity with respect to the distribution parameters of inputs [2]. The main drawback of the derivative based SA is that it depends on the choice of the nominal point, but it is generally mathematically simple and straightforward [3].

Global SA studies how the uncertainty in the output of a computational model can be decomposed according to the input sources of uncertainty [4]. Contrary to the local SA, global SA explores the whole range of uncertainty of the model inputs by letting them vary simultaneously [5]. At present, a number of global sensitivity indices have been suggested, e.g. Helton and Saltelli [6,7] proposed the nonparametric sensitivity indices (input–output correlation), Sobol, Iman and Saltelli [7–9] proposed the variance based sensitivity indices, Chun, Liu and Borgonovo [10,11] proposed moment independent sensitivity indices. In this work, we mainly investigate the variance based sensitivity indices which have been applied to design under uncertainty problems and are capable of identifying the contributions of any random variable. However, the variance based sensitivity indices are far more computationally demanding [12].

It is noticed that in the classical variance based SA, the influences of distribution parameters are not involved. If the variation of a distribution parameter can lead to a considerable change to the variance, the computational results of the variance based SA will be vulnerable and less reliable. Thus, it is significant to identify how the distribution parameters influence the variance. To do this, this work employs the derivative based SA by defining the derivatives of the variance with respect to the distribution parameters. According to the Sobol′s variance decomposition theory [8], the influence of the distribution parameters on the variance can be transmitted by various orders variance contributions. Thus, the influence of the distribution parameter on the various orders variance contribution is investigated. Based on that, this work presents the derivative based main and total sensitivity indices, which can be used to identify the influence of the distribution parameter on the main and total variance contribution.

By employing the kernel function to simplify the derivatives of the variance contribution with respect to the distribution parameters, the proposed main and total sensitivity indices can be

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computed easily and can be seen as the "by-product" of the classical variance based SA without additionally computational cost. Since a large number of methods have been used to compute the Sobol′s variance based sensitivity indices, such as the quasi-Monte Carlo method [12], FAST method [13], the meta-model based method [14], the sparse grid integration method [15] etc, and the sparse grid integration (SGI) method has been proved to be a high efficiency one in reference [16], this work would employ the SGI method to compute the proposed main and total indices.

The remainder of this work is organized as follows: Section 2 gives a brief review of the variance based SA. Section 3 gives the decomposition of the derivative based variance sensitivity and proposes the corresponding main and the total sensitivity indices. In Section4, the kernel function is used to simplify the derivatives of the variance contributions with respect to the distribution parameters and some discussions are given. The spares grid integration method is employed in Section 5 to compute the proposed main and total sensitivity indices. In Section 6, two numerical examples are first given to validate the rationality of the proposed sensitivity indices and the accuracy of the SGI method. Then a simple cantilever beam with explicit response and a ten-bar structure with implicit response are analyzed. Finally, some conclusions are drawn in Section 7.

2. Brief review of the classical variance based sensitivity analysis

Consider a square integrable function $Y=g(\boldsymbol{X})$ defined in the hypercube H^n, where $\boldsymbol{X}=(X_1,X_2,\ldots,X_n)$ are n independent inputs. The method of variance based sensitivity indices developed by Sobol is based on ANOVA decomposition [8] and there exists the following unique decomposition:

$$g(\boldsymbol{X})=g_0+\sum_{i=1}^{n} g_i(X_i)+\sum_{1\le i<j\le n}^{n} g_{ij}(X_i,X_j)+\cdots+g_{1,2,\ldots,n}(X_1,X_2,\ldots,X_n) \tag{1}$$

where

$$\begin{aligned} g_0&=E(Y)\\ g_i&=E(Y|X_i)-g_0\\ Y_{ij}&=E(Y|X_i,X_j)-g_i-g_j-g_0 \end{aligned} \tag{2}$$

$E(Y)$ is the expectation of output, $E(Y|\bullet)$ is the conditional expectation of output and the higher-order items can be obtained similarly.

The basic idea of variance based SA is to decompose the model into terms of increasing dimensionality as in Eq. (1). The variance of the output variable Y can thus be decomposed into:

$$V=\sum_{i=1}^{n} V_i+\sum_{1\le i<j\le n}^{n} V_{ij}+\cdots+V_{1,2,\ldots,n} \tag{3}$$

where V is the total variance, and

$$\begin{aligned} V_i&=V(Y_i)=V(E(Y|X_i))\\ V_{ij}&=V(Y_{ij})=V(E(Y|X_i,X_j))-V(E(Y|X_i))-V(E(Y|X_j)) \end{aligned} \tag{4}$$

are the first-order and the second-order variance contributions respectively.

Then, the total variance contribution can be represented as

$$VT_i=V_i+\textstyle\sum_{j\ne i}V_{ij}+\cdots+V_{1,2,\ldots,n}=V-V(E(Y|\boldsymbol{X}_{-i}))=V-V_{-i} \tag{5}$$

where $\boldsymbol{X}_{-i}=(X_1,\ldots,X_{i-1},X_{i+1},\ldots,X_n)$ is the vector of all inputs except X_i.

Dividing Eq. (3) by the total variance V yields

$$1=\sum_{i=1}^{n} S_i+\sum_{1\le i<j\le n}^{n} S_{ij}+\cdots+S_{1,2,\ldots,n} \tag{6}$$

where

$$S_i=\frac{V(E(Y|X_i))}{V(Y)} \tag{7}$$

is defined as the main effect for input X_i, and

$$ST_i=S_i+\textstyle\sum_{j\ne i}S_{ij}+\cdots+S_{1,\ldots,n}=1-\dfrac{V(E(Y|\boldsymbol{X}_{-i}))}{V(Y)} \tag{8}$$

is defined as the total effect for input X_i.

The main effect S_i indicates how much one could reduce, on average, the output variance if X_i could be fixed. By definition, the total effect ST_i is greater than S_i, or equal to S_i in case that X_i is not involved in any interaction with other inputs. Thus, the total effect ST_i takes into account both the main effect of input X_i and the interaction effects of it with other inputs. Especially, $ST_i=0$ implies that X_i is non-influential and can be fixed anywhere in its distribution without affecting the variance of the output and $ST_i=S_i$ holds for a purely additive model [4].

3. The derivative based variance sensitivity analysis

3.1. Definition

As discussed in Section 1, the variance sensitivity can be defined as the derivative of the total variance with respect to the distribution parameter. Without any lack of generality, we suppose that each input only depends on one distribution parameter in order to simplify the notation in the following. For the influential distribution parameter, it is significant to identify how it influences the variance. Because the Sobol's variance decomposition is orthogonal, then the variance sensitivity can also be decomposed as

$$\frac{\partial V}{\partial\theta_k}=\sum_{i=1}^{n}\frac{\partial V_i}{\partial\theta_k}+\sum_{1\le i<j\le n}^{n}\frac{\partial V_{ij}}{\partial\theta_k}+\cdots+\frac{\partial V_{1,2,\ldots,n}}{\partial\theta_k} \tag{9}$$

where $\theta_k(k=1,2,\ldots,n)$ is the distribution parameter of input $X_k(k=1,2,\ldots,n)$.

$$\begin{aligned} \frac{\partial V_i}{\partial\theta_k}&=\frac{\partial V(E(Y|X_i))}{\partial\theta_k}\\ \frac{\partial V_{ij}}{\partial\theta_k}&=\frac{\partial V(E(Y|X_i,X_j))}{\partial\theta_k}-\frac{\partial V(E(Y|X_i))}{\partial\theta_k}-\frac{\partial V(E(Y|X_j))}{\partial\theta_k} \end{aligned} \tag{10}$$

are the derivatives of the first-order and the second-order variance contributions with respect to the distribution parameter θ_k, respectively.

It is noticed that if θ_k is not the distribution parameter of input X_i, it still influences the various orders variance contributions of input X_i. As shown in Fig. 1, the first-order variance contribution of input X_i not only includes the uncertainty of input X_i, but also includes that of X_k.

Eq. (9) reflects that the influence of distribution parameter θ_k on the variance can be transmitted by the various orders of variance contributions which can also be shown in Fig. 2.

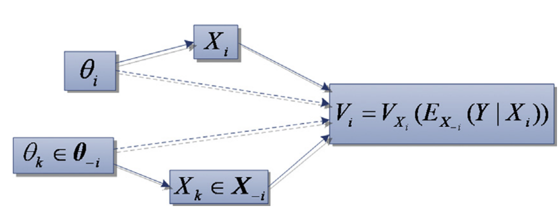

Fig. 1. Influence of the distribution parameter on the first-order variance contribution.