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Dissolving multi-partnerships efficiently☆

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ABSTRACT

I study a market where agents with unit demand jointly own heterogeneous goods. In this market, the existence of an efficient, incentive compatible, individually rational, and budget balanced mechanism depends on the shares of the agents. I characterize the set of shares for which having such a mechanism is possible. This set includes the symmetric allocation and excludes the allocation in which every agent owns a separate good.

1. Introduction

How should a condominium association assign units to its member families after the construction is completed? In practice, a random order of families is drawn. According to the draw, each family then chooses from the remaining units until everyone has chosen a unit. This procedure is called the *random serial dictatorship*. Now, suppose that the second family is willing to pay a handsome subsidy to get the unit chosen by the first family. These two families could both benefit if they swapped their houses and the second family paid some money to the first family. Hence, the serial random dictatorship is not *efficient*. Is there an efficient mechanism to allocate these units? Similarly, suppose that a group of college students rents an apartment. How should they allocate the rooms and divide up the rent? Is there an efficient procedure to solve this problem?[1]

I consider a class of economic problems where there are n agents and n goods encompassing the condominium association problem and the room assignment–rent division problem. Each agent initially owns shares of every good that add up to one. Moreover, the final allocation is constrained so that each agent gets one good. In the condominium example, one can think that each member of the association has a claim to $1/n$th of each unit, but each unit needs to be allocated to only one agent. In this environment, I study whether an efficient, individually rational, incentive compatible, and budget balanced mechanism exists to 'dissolve' the multi-partnership.

I characterize the set of initial shares for which having a mechanism with the required properties is possible. I further show that this set is convex. This result is similar to that of Cramton et al. (1987) (CGK) even though the setups are different. To be more explicit, in CGK there is one item jointly owned by many agents. In that setup, they characterize the set of initial shares when efficient partnership dissolution is possible and they provide a mechanism to dissolve the partnership. If there are more goods without any restrictions on how many goods agents own initially or finally, their analysis can still be used. In particular, if efficient partnership dissolution is possible, then their mechanism can be used good-by-good to dissolve all the partnerships. In contrast, the current model requires each agent to own one unit initially and finally. This is motivated by the aforementioned examples. Because of this difference, the results in CGK do not apply to the current problem.

A special case is when all agents have the same shares, for which I show the existence of a mechanism with the required properties. This is the case in the condominium association and the room assignment–rent division problem. Another special case of the setup is when each agent owns one good individually. This can be thought of as a housing market where people own one house but want to relocate. When $n = 2$ this case can be modified such that it coincides with the setup in Myerson and Satterthwaite (1983) who study a model with one buyer and one seller with a good. They show that if agents have overlapping values, then it is impossible to trade efficiently. I extend their classic impossibility result for general n.

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[1] See Abdulkadiroğlu and Sönmez (1998) for a characterization of the random serial dictatorship and Abdulkadiroğlu et al. (2004) for a solution of the room assignment–rent division problem which can be manipulated by the agents.

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The method used in this paper builds on the results of two lines of research. One is the revenue equivalence condition in the presence of multidimensional signals: Krishna and Maenner (2001) and Jehiel et al. (1999) prove it for general cases which admit my setup as a special case. The second line of research, using the revenue equivalence condition, gives a necessary and sufficient condition for the implementation of an allocation rule; see Makowski and Mezzetti (1994), Krishna and Perry (1998), Williams (1999) and Fieseler et al. (2003) for example. I use the same technique in this paper. The execution of the technique is not straightforward, because in a setup with multiple dimensional types one needs to calculate which type of an agent gives that agent the minimum expected utility at the interim stage when each agent only knows their type. I provide a characterization by proving bijectivity of a carefully chosen function. Moreover, even after these types are calculated one needs to verify an inequality involving them which is, *a priori*, not trivial.

In an independent work, Gershkov and Schweinzer (2010) analyze a queueing problem where agents have the same value for a service provided with differing waiting costs. They show that if the access rights to the service is a random order, then the efficient trading is possible whereas a deterministic order makes it impossible. Although their problem is different, the intuition for their results and mine are similar. Subsequently, Segal and Whinston (2011) study incentive compatible mechanisms in a general model. They show that if the expectation of the equilibrium allocation is equal to the status quo allocation then individual rationality is satisfied. In particular, if the allocation rule is efficient, then efficiency is attained along with individual rationality and incentive compatibility. In the current model, this translates to the positive result that efficient trading is possible when initial shares are symmetric. However, in contrast to the current study, they do not analyze what happens when the status quo is different from the expected equilibrium allocation.

The rest of the paper is organized as follows. Section 2 introduces the model. Section 3 states a necessary and sufficient condition to have a mechanism with the required properties. Section 4 analyzes this condition for different initial shares. The last section summarizes the paper and outlines future research directions.

2. Model

Consider a set of agents $A = \{1, 2, \ldots, n\}$, and a set of goods $G = \{1, 2, \ldots, n\}$. Each agent i has a valuation vector $\theta_i \in [\underline{v}, \overline{v}]^n$ where θ_{ij} is the value that agent i assigns to good j. For any i, all θ_{ij}s are drawn independently from a distribution F_i with support $[\underline{v}, \overline{v}]$ and continuous positive density f_i. Moreover, for $i \neq i'$, θ_i is independent of $\theta_{i'}$. Finally, each agent's valuation vector θ_i is known privately and it is common knowledge that θ_i is drawn from F_i for all i.

The goods are initially owned by the agents collectively. Agent i owns α_{ij} shares of good j. Let $\alpha_i \in \mathbb{R}^n_+$ be the vector of initial shares owned by agent i and $\alpha = (\alpha_1; \alpha_2; \ldots; \alpha_n)$ be the matrix of shares which is commonly known by the agents. Every good is owned completely by the set of all agents, so $\sum \alpha_i = (1, 1, \ldots, 1)$.[2] Initially, every agent owns one good in total, $\sum_j \alpha_{ij} = 1$ for all i.

I assume that each agent's utility is linear in money and assets. To be more specific, an agent with valuation vector θ_i, shares s_i, and money m_i has utility $\theta_i \cdot s_i + m_i$.

By the revelation principle, it is sufficient to consider direct revelation mechanisms (DRM) for analyzing the existence of a mechanism with the required properties. In a DRM $\langle p, t \rangle$, all agents report their own types θ_i to get $p_i(\theta)$ share of the goods and a monetary transfer $t_i(\theta; \alpha)$.[3] Let $P_i(\theta_i) = \mathbb{E}_{\theta_{-i}}[p_i(\theta)]$ be the expected shares that agent i is going to get from participation and $T_i(\theta_i; \alpha) = \mathbb{E}_{\theta_{-i}}[t_i(\theta; \alpha)]$ be the expected transfers. Therefore, in the interim stage when agent i only knows θ_i their expected gain from participating in the mechanism is $U_i(\theta_i; \alpha) = \theta_i \cdot (P_i(\theta_i) - \alpha_i) + T_i(\theta_i; \alpha)$. If agent i misreports their type to be θ_i' then their utility is $\theta_i \cdot (P_i(\theta_i') - \alpha_i) + T_i(\theta_i'; \alpha)$.

A mechanism $\langle p, t \rangle$ *implements* allocation rule p if truth-telling is a Bayesian–Nash equilibrium. Such a mechanism is called (interim) *incentive compatible* (IIC) and satisfies the following inequality:

$$U_i(\theta_i; \alpha) \geq \theta_i \cdot (P_i(\theta_i') - \alpha_i) + T_i(\theta_i'; \alpha) \quad \text{for all } \theta_i \text{ and } \theta_i'.$$

A mechanism $\langle p, t \rangle$ which gives non-negative expected utility in the interim stage to all the agents is called (interim) *individually rational* (IIR): $U_i(\theta_i; \alpha) \geq 0$ for all θ_i. Furthermore, this mechanism is called *ex-ante budget balanced* (EABB) if the sum of expected transfers is zero, that is $\mathbb{E}_\theta[\sum t_i(\theta; \alpha)] = 0$. Similarly, it is called (ex post) *budget balanced* (BB) if $\sum t_i(\theta; \alpha) = 0$ for all θ. It is *efficient* (EF) if $\sum \theta_i \cdot p_i(\theta) \geq \sum \theta_i \cdot p_i'(\theta)$ for all allocation rules p' and for all θ. In this case p is also called the *efficient allocation rule*. Note that efficiency is a condition on the allocation rule and pins it down generically. Moreover, for generic θ, the efficient allocation rule assigns one separate good to each agent. Furthermore, the efficient allocation rule satisfies the following.

Fact 1. *Suppose that p is the efficient allocation rule. Then $p_i(\theta) = p_i(\theta + c\mathbf{1}^\top)$ for all i and almost all θ where $c \in \mathbb{R}^n$.*[4]

This fact follows directly from the definition. If $\sum \theta_i \cdot p_i(\theta) \geq \sum \theta_i \cdot p_i'(\theta)$ for all p', then $\sum (\theta_i + c_i\mathbf{1}) \cdot p_i(\theta) \geq \sum (\theta_i + c_i\mathbf{1}) \cdot p_i'(\theta)$ for all p'.

Since I concentrate on efficient mechanisms, the only choice variable is the transfer function.

3. The existence condition

In this section I provide a necessary and sufficient condition for a mechanism to satisfy the properties listed. First I state a revenue equivalence theorem below, from which it follows that any two EF and IIC mechanisms are interim payoff equivalent up to a constant. Therefore, to analyze the existence of such a mechanism which also satisfies EABB and IIR, it is sufficient to use a VCG mechanism which is EF and IIC. Hence, I derive a necessary and sufficient condition to have an EF, IIC, IIR, and EABB condition which is also implementable in dominant-strategies since VCG mechanisms make it a dominant strategy for agents to report their types truthfully.[5] By using the transformation in Arrow (1979) and d'Aspremont and Gérard-Varet (1979) (Arrow-AGV transformation), EABB can be strengthened to BB at the expense of wakening dominant-strategy incentive compatibility to IIC.

3.1. Some preliminary results

To begin, I give the revenue equivalence condition for the setup.

[2] Summations are always over i unless stated otherwise.

[3] Given an allocation problem, α is a constant which is commonly known by all the agents. Even though agents do not announce α in the DRM, the payments can depend on it. To analyze the problem for different values of α, it is explicitly written in the transfer function.

[4] Here, $\mathbf{1}$ is the n-dimensional vector consisting of ones and $c\mathbf{1}^\top$ is the matrix multiplication of c with the transpose of $\mathbf{1}$.

[5] *Dominant strategy incentive compatibility* for a DRM requires that each agent prefers to report her type truthfully regardless of what other agents report.