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# An economic lot-size model with non-linear holding cost hinging on time and quantity

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ABSTRACT

This paper develops an economic lot size inventory model where the demand rate depends on the stock level and the cumulative holding cost is non-linear on both the quantity and the time they are stored. More concretely, it is supposed that the demand rate is a concave potential function of the inventory level and the holding cost is potential on both time and quantity. Moreover, shortages are not allowed. A general procedure to determine the optimal lot size and the maximum inventory profit is developed. Also, some results about the profitability of the inventory system are presented. This work extends several inventory models previously considered in the literature. Finally, numerical examples, which help us to understand the theoretical results, are also given.



## 1. Introduction

Since Harris presented his EOQ model in 1913, many efforts have been made to adjust their assumptions to more realistic situations in inventory management. As is known, in order to obtain a simple mathematical model, Harris assumed that shortage was not allowed and both demand rate and holding costs per unit and per unit time were constant. Thus, the hypotheses of the classic model ensured that the incomes were independent of the stock level and, therefore, the optimal lot size was obtained by minimizing the sum of the carrying cost and the ordering cost per unit of time. However, as is also well-known, empirical evidence shows that these hypotheses are an excessive simplification of reality. So, many models developed later worked with the idea of relaxing these assumptions. In what follows, we will focus on inventory models which maintain the assumption of not allowing shortage, but the other hypotheses have been revised; that is, we do not consider that both demand rate and holding cost per unit and per unit of time are constant.

Over time, some practitioners and researchers have found that an increase in exposed merchandise may bring about increased sales of some items. So, Wolfe (1968) presented empirical evidence of this relationship, showing that the sales of style merchandise, such as women's dresses or sports clothes, are proportional to the amount of displayed inventory. After this observation, mathematical models for inventory systems captured this idea by regarding the use of stock level dependent demand rate in their formulation. As a starting point, Baker and Urban (1988) defined a model in which the demand rate was supposed to be a concave potential function of the inventory level. This new approach to the problem brought about the loss of independence between revenues and total inventory costs, because a larger order size results not only in higher inventory costs but also in higher revenues. Therefore, in this situation, besides the ordering and holding costs, it becomes necessary to consider a third component for the inventory system: the gross profit from the sale of the item (the difference between the selling price and the purchasing cost). Moreover, in this case, the principal objective should be to maximize the profit of the inventory system per unit time instead of minimizing the total inventory cost per unit time. For this reason, Baker and Urban (1988) raised the issue from the perspective of maximizing the profit per unit time. Since then, many papers have appeared considering stock dependent demand rate, but some approach the problem from the perspective of minimizing the total inventory cost per unit time. Urban (2005) provides a detailed overview of the related literature published up to that date. Dye and Ouyang (2005) proposed an inventory model with a demand rate linearly dependent on the stock level and shortages with a time-proportional backlogging rate, which is an extension of the model developed by Padmanabhan and Vrat (1995). Teng and Chang (2005) considered an EPQ model with maximizing profits where the demand rate is simultaneously dependent on the stock level and the selling price but with a ceiling. Also, You (2005) and Roy (2008) analyzed EOQ models with price dependent demand.

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Another issue presented by inventory managers was that, sometimes, the holding cost per unit of item is not proportional to the time held in stock, and/or the holding cost per unit time is also not proportional to the amount held in inventory. This question raised the need to consider other structures for the holding cost in the mathematical models of inventory systems. Under the hypothesis of a constant demand rate, Naddor (1982) proposed three inventory models based on the holding cost function per cycle for those situations. The first model considered that the holding cost per cycle was non-linear with time, but linear with the quantity of items (the perishable-goods system), the second one assumed that the holding cost per cycle was non-linear with the quantity of items, but linear with time (the expensive-storage system) and the third model supposed that the holding cost per cycle was non-linear with both time and quantity of items (the general system). In this environment, Weiss (1982) introduced a deterministic model with constant demand rate, supposing that the holding cost per unit of product was a convex potential function of the time in stock. This situation can occur, for example, in the storage of perishable items such as food products. In this case, the longer the products are kept in storage, the more sophisticated the storage facilities and services are needed, and therefore, the higher the holding cost. In fact, this model proved to be equivalent to the perishable-goods system cited above, as noted by Ferguson et al. (2007). Later on, Goh (1994) generalized this situation to the case of stock-dependent demand rate, minimizing the total inventory cost per unit time. Alfares (2007) considered this same situation with two types of discontinuous step functions. Urban (2008) extended Alfares' work by using a profit maximization objective. Recently, Pando et al. (2012a) have studied the previous model of Goh (1994), but from the perspective of maximizing the profit.

Inventory models with holding cost rate per unit time non-linear in the stock level are scarcer. Nevertheless, this situation can be encountered in real inventories when the value of the item is very high and many precautionary steps are to be taken to ensure its safety and quality. This occurs, for example, when carrying luxury items like expensive jewelry and designer watches. From the perspective of minimizing the total inventory cost per unit of time and using a stock-dependent demand rate, Goh (1994) supposed that the holding cost rate per unit of time was a convex potential function of the quantity of items held in stock. Later on, Giri and Chaudhuri (1998) extended this model to the case of deteriorating items with a small constant fraction of deterioration. Berman and Perry (2006) also worked with stock-dependent holding cost and two types of demand rate functions. Scarpello and Ritelli (2008) gave some theoretical results for EOQ models when the holding cost grows with the stock level. Recently, Pando et al. (2012b) have analyzed one of the models of Goh (1994) from the perspective of maximizing the profit.

To the best of our knowledge, the work of Naddor (1982) is the only paper dedicated to study inventory models for the situation in which the holding cost is non-linear with both time and quantity. This structure of the holding cost function is adequate for the representation of some real life situations. This occurs, for example, in the storage of very expensive products (then many precautionary steps are to be taken to ensure their safety) which deteriorates over time (so more sophisticated storage facilities and services are needed). However, in the work of Naddor (1982), the demand rate was considered constant.

According to the previous comments, the objective of this paper is to study an inventory system with stock dependent demand rate and a structure of holding cost non-linear in both time and stock level from the perspective of maximizing profits per unit time. Thus, the main difference between our model and the one in Naddor (1982) is the assumption about the demand rate. So, instead of a constant demand rate, we consider the case where the demand rate is a concave potential function of the inventory level. Moreover, we develop the necessary theory to calculate the holding cost per cycle when it depends non-linearly on both time and stock level.

The structure of this paper is as follows. Section 2 presents the assumptions and notation to be used throughout the paper. Section 3 deals with the mathematical formulation of the inventory model, the procedure to obtain the optimal policy and the main theoretical results. In Section 4, we provide some numerical examples to illustrate the theoretical results and show the difference between the optimal solutions for the problems of maximum profit and minimum cost. Finally, conclusions are drawn in Section 5.

## 2. Assumptions and notation

The inventory system studied in this paper is developed on the basis of the following assumptions. The inventory is continuously reviewed and the planning horizon is infinite. The item is a single product and no shortage is allowed. The replenishment is instantaneous when the stock is depleted and the lead time is zero. The demand rate is a known function of the inventory level. The order cost is fixed, regardless of the lot size. The unit purchasing cost and the selling price are known and constant. The cumulative holding cost for $x$ units of product that have been stored during $t$ units of time is a potential function of $t$ and $x$. Thus, this function may not be linear in any of these two variables.

Table 1 summarizes the notation used in this paper.

We suppose that the demand rate is described by the potential function of the inventory level

$$D(t) = \lambda[I(t)]^{\beta} \tag{1}$$

with $\lambda > 0$ and $0 \le \beta < 1$. With this functional form, as the inventory level decreases, so does the demand rate. Thus, at the beginning of a cycle, the inventory level decreases rapidly because the demand is bigger at a high level of stock. At time $t = 0$ the inventory level and the demand rate are at their highest level. As more inventory is depleted, the rate of decrease of the stock level slows down. The elasticity of the demand rate with respect to the stock level $\beta$ represents the relative change in demand rate with respect to the corresponding relative change in the stock level (that is, $\beta = (\partial D(t)/\partial I(t))/(D(t)/I(t))$). Note that, if the demand is inelastic (that is, $\beta = 0$), we have $D(t) = \lambda$ and the model reverts to the well-known inventory model with constant demand rate.

Furthermore, it is supposed that the cumulative holding cost for $x$ items stored during $t$ units of time is given by the following

**Table 1**
List of notation.

| | |
|---|---|
| $q$ | Order quantity or lot size per cycle ($>0$, decision variable) |
| $T$ | Length of the inventory cycle ($>0$) |
| $t$ | Time spent in inventory ($\le T$) |
| $I(t)$ | Inventory level at time $t$ ($\le q$) |
| $K$ | Ordering cost per order ($>0$) |
| $p$ | Unit purchasing cost ($>0$) |
| $s$ | Unit selling price ($\ge p$) |
| $H(t,x)$ | Cumulative holding cost for $x$ items stored during $t$ units of time ($>0$) |
| $h$ | Scaling constant for holding cost ($>0$) |
| $\gamma_1$ | Elasticity of the holding cost with respect to time ($\ge 1$) |
| $\gamma_2$ | Elasticity of the holding cost with respect to the stock level ($\ge 1$) |
| $D(t)$ | Demand rate at time $t$ |
| $\lambda$ | Scaling constant of demand rate ($>0$) |
| $\beta$ | Elasticity of demand rate with respect to the stock level ($0 \le \beta < 1$) |
| $\alpha$ | Auxiliary parameter, $\alpha = 1-\beta$ ($0 < \alpha \le 1$) |
| $\xi$ | Auxiliary parameter, $\xi = \alpha\gamma_1 + \gamma_2$ ($>1$) |