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# The dual decomposition of aggregation functions and its application in welfare economics

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**Abstract**

In this paper, we review the role of self-duality in the theory of aggregation functions, the dual decomposition of aggregation functions into a self-dual core and an anti-self-dual remainder, and some applications to welfare, inequality, and poverty measures.


*Keywords:* Aggregation functions; Self-duality; Dual decomposition; Inequality; Welfare; Poverty

## 1. Introduction

In the context of aggregation functions, self-duality is an important property (see Beliakov et al. [3] and Grabisch et al. [16]). Self-dual aggregation functions satisfy $A(\mathbf{1}-\boldsymbol{x}) = 1 - A(\boldsymbol{x})$ for every $\boldsymbol{x} \in [0,1]^n$. In other words, the aggregate value of the transformed inputs coincides with the transformed aggregate value of the original inputs. This means that the aggregation function is unbiased relatively to the higher or lower value of its inputs.

In the aggregation of reciprocal preference relations, for instance, self-duality ensures the reciprocity of the aggregate preference relation (see García-Lapresta and Llamazares [12]).

Silvert [24] introduced *symmetric sums*, a class of self-dual aggregation functions with two variables, within the context of his characterization of self-duality (see also Dubois and Prade [8] and Calvo et al. [6, p. 32]).

García-Lapresta and Marques Pereira [13,14] proposed a method that associates a self-dual aggregation function to any aggregation function. This method improves the one given by Silvert [24] in a number of ways (see García-Lapresta and Marques Pereira [14, Sect. 4]).

Maes et al. [20] provide a characterization of self-dual aggregation functions which generalizes those given by Silvert [24] and García-Lapresta and Marques Pereira [14]. In turn, Maes and De Baets [19] merge self-dual and commutative binary aggregation functions in a single functional equation.

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The paper is organized as follows. Section 2 reviews basic notions regarding aggregation functions and their dual decomposition, with a particular focus on exponential means and OWA functions. Section 3 discusses some applications of the dual decomposition to welfare economics, and Section 4 contains some concluding remarks.

## 2. Aggregation functions

We now present notation and basic definitions regarding aggregation functions on $[0,1]^n$, with $n \in \mathbb{N}$ and $n \geq 2$ throughout the text. For further details the interested reader is referred to Fodor and Roubens [10], Calvo et al. [6], Beliakov et al. [3], García-Lapresta and Marques Pereira [14] and Grabisch et al. [16].

Vectors in $[0,1]^n$ are denoted as $\boldsymbol{x} = (x_1, \ldots, x_n)$, $\mathbf{0} = (0, \ldots, 0)$, $\mathbf{1} = (1, \ldots, 1)$. Accordingly, for every $x \in [0,1]$, we have $x \cdot \mathbf{1} = (x, \ldots, x)$. Given $\boldsymbol{x}, \boldsymbol{y} \in [0,1]^n$, by $\boldsymbol{x} \geq \boldsymbol{y}$ we mean $x_i \geq y_i$ for every $i \in \{1, \ldots, n\}$, and by $\boldsymbol{x} > \boldsymbol{y}$ we mean $\boldsymbol{x} \geq \boldsymbol{y}$ and $\boldsymbol{x} \neq \boldsymbol{y}$. Given $\boldsymbol{x} \in [0,1]^n$, the increasing and decreasing reorderings of the coordinates of $\boldsymbol{x}$ are indicated as $x_{(1)} \leq \cdots \leq x_{(n)}$ and $x_{[1]} \geq \cdots \geq x_{[n]}$, respectively. In particular, $x_{(1)} = \min\{x_1, \ldots, x_n\} = x_{[n]}$ and $x_{(n)} = \max\{x_1, \ldots, x_n\} = x_{[1]}$. Clearly, $x_{[k]} = x_{(n-k+1)}$ for every $k \in \{1, \ldots, n\}$. In general, given a permutation $\sigma$ on $\{1, \ldots, n\}$, we denote $\boldsymbol{x}_\sigma = (x_{\sigma(1)}, \ldots, x_{\sigma(n)})$. The arithmetic mean of $\boldsymbol{x}$ is denoted by $\mu(\boldsymbol{x})$.

**Definition 1.** Let $A : [0,1]^n \longrightarrow \mathbb{R}$ be a function.

1. $A$ is *idempotent* if for every $x \in [0,1]$ it holds that $A(x \cdot \mathbf{1}) = x$.
2. $A$ is *symmetric* if for every permutation $\sigma$ on $\{1, \ldots, n\}$ and every $\boldsymbol{x} \in [0,1]^n$ it holds that $A(\boldsymbol{x}_\sigma) = A(\boldsymbol{x})$.
3. $A$ is *monotonic* if for all $\boldsymbol{x}, \boldsymbol{y} \in [0,1]^n$ it holds that $\boldsymbol{x} \geq \boldsymbol{y} \Rightarrow A(\boldsymbol{x}) \geq A(\boldsymbol{y})$.
4. $A$ is *strictly monotonic* if for all $\boldsymbol{x}, \boldsymbol{y} \in [0,1]^n$ it holds that $\boldsymbol{x} > \boldsymbol{y} \Rightarrow A(\boldsymbol{x}) > A(\boldsymbol{y})$.
5. $A$ is *compensative* (or *internal*) if for every $\boldsymbol{x} \in [0,1]^n$ it holds that $x_{(1)} \leq A(\boldsymbol{x}) \leq x_{(n)}$.
6. $A$ is *self-dual* if for every $\boldsymbol{x} \in [0,1]^n$ it holds that $A(\mathbf{1} - \boldsymbol{x}) = 1 - A(\boldsymbol{x})$.
7. $A$ is *anti-self-dual* if for every $\boldsymbol{x} \in [0,1]^n$ it holds that $A(\mathbf{1} - \boldsymbol{x}) = A(\boldsymbol{x})$.
8. $A$ is *invariant for translations* if for every $\boldsymbol{x} \in [0,1]^n$ it holds that $A(\boldsymbol{x} + t \cdot \mathbf{1}) = A(\boldsymbol{x})$ for every $t \in \mathbb{R}$ such that $\boldsymbol{x} + t \cdot \mathbf{1} \in [0,1]^n$.
9. $A$ is *stable for translations* (or *shift-invariant*) if for every $\boldsymbol{x} \in [0,1]^n$ it holds that $A(\boldsymbol{x} + t \cdot \mathbf{1}) = A(\boldsymbol{x}) + t$ for every $t \in \mathbb{R}$ such that $\boldsymbol{x} + t \cdot \mathbf{1} \in [0,1]^n$.

**Definition 2.** Let $\left(A^{(k)}\right)_{k \in \mathbb{N}}$ be a sequence of functions, with $A^{(k)} : [0,1]^k \longrightarrow \mathbb{R}$ and $A^{(1)}(x) = x$ for every $x \in [0,1]$. $\left(A^{(k)}\right)_{k \in \mathbb{N}}$ is *invariant for replications* (or *strongly idempotent*) if for all $\boldsymbol{x} \in [0,1]^n$ and any number of replications $m \in \mathbb{N}$ of $\boldsymbol{x}$ it holds that

$$A^{(mn)}(\overbrace{\boldsymbol{x}, \ldots, \boldsymbol{x}}^{m}) = A^{(n)}(\boldsymbol{x}).$$

**Definition 3.** Consider the binary relation $\succcurlyeq$ on $[0,1]^n$, defined as

$$\boldsymbol{x} \succcurlyeq \boldsymbol{y} \Leftrightarrow \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i \text{ and } \sum_{i=1}^{k} x_{(i)} \leq \sum_{i=1}^{k} y_{(i)},$$

for every $k \in \{1, \ldots, n-1\}$.

1. A function $A : [0,1]^n \longrightarrow [0,1]$ is *S-convex* if for all $\boldsymbol{x}, \boldsymbol{y} \in [0,1]^n$:

$$\boldsymbol{x} \succcurlyeq \boldsymbol{y} \Rightarrow A(\boldsymbol{x}) \geq A(\boldsymbol{y}).$$

2. A function $A : [0,1]^n \longrightarrow [0,1]$ is *strictly S-convex* if for all $\boldsymbol{x}, \boldsymbol{y} \in [0,1]^n$:

$$\boldsymbol{x} \succ \boldsymbol{y} \Rightarrow A(\boldsymbol{x}) > A(\boldsymbol{y}),$$

where $\boldsymbol{x} \succ \boldsymbol{y}$ means $\boldsymbol{x} \succcurlyeq \boldsymbol{y}$ and $\boldsymbol{x} \neq \boldsymbol{y}$.