



The fundamental theorem of asset pricing in the presence of bid-ask and interest rate spreads[☆]

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ABSTRACT

We establish the fundamental theorem of asset pricing to a model with proportional transaction costs on trading in shares and different interest rates for borrowing and lending of cash. We show that such a model is free of arbitrage if and only if one can embed in it a friction-free model that is itself free of arbitrage, i.e. if there exists an artificial friction-free price for the stock between its bid and ask prices and an artificial interest rate between the borrowing and lending interest rates such that, if one discounts this stock price by this interest rate, then the resulting process is a martingale under some equivalent probability measure.

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The fundamental theorem of asset pricing characterises models of financial markets without arbitrage or free lunch, i.e. the making of risk-free profit without initial investment. It is well known that a classical friction-free model containing a risky stock and a bank account admits no arbitrage if and only if there exists a probability measure on the model under which the stock price, discounted by the interest rate on the bank account, is a martingale. This result was first established for discrete-time models by Harrison and Pliska (1981) (finite state space) and Dalang et al. (1990) (infinite state space), and for continuous-time models by Delbaen and Schachermayer (1994, 1998).

The main contribution of this paper is to extend this result in two directions, to simultaneously include proportional transaction costs in the form of bid-ask spreads on stock prices and different interest rates for borrowing and lending of cash. We show that such a model is free of arbitrage if and only if one can embed in it a friction-free model that is itself free of arbitrage, in the sense that there exists an artificial friction-free price for the stock between its bid and ask prices and an artificial interest rate between the borrowing and lending interest rates such that, if one discounts

this stock price by this interest rate, then the resulting process is a martingale under some non-degenerate probability measure.

The fundamental theorem of asset pricing is well established in models with proportional transaction costs where borrowing and lending takes place at the same rate. Generally speaking, such a model is free of arbitrage if and only if there exists an artificial stock price process taking its values between the bid and ask prices of the stock, and an equivalent probability measure under which the discounted value of this process is a martingale. This result was first established in a continuous-time setting by Jouini and Kallal (1995b, Theorem 3.2). In discrete time, Kabanov and Stricker (2001, Theorem 2 and Section 5) and Ortu (2001, Theorem 3) extended the result of Harrison and Pliska (1981) to include proportional transaction costs, and Zhang et al. (2002, Theorem 3.1), Kabanov et al. (2002, Theorem 1) and Schachermayer (2004, Theorem 1.7) did the same for the work of Dalang et al. (1990).

Different borrowing and lending rates are most commonly viewed as a special case of models with convex constraints on portfolio positions, more precisely by viewing cash holdings as two separate assets, one of which may only have a non-negative balance and accrues interest at the lending rate, and another that may only have a non-positive balance and accrues interest at the borrowing rate. Jouini and Kallal (1995a, Theorem 2.1) was the first to establish a version of the fundamental theorem of asset pricing in a (continuous-time) model with constraints of this type. In

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discrete time, Schürger (1996, Theorem 2.4) extended the work of Dalang et al. (1990) to establish a version of the fundamental theorem of asset pricing in a model with portfolio constraints of this kind, but under the assumption that the model contains a risk-free bank account, with a single (deterministic) interest rate for borrowing and lending; this was extended by Pham and Touzi (1999, Theorem 4.2) to allow general (deterministic) convex cone constraints on stock holdings. Evstigneev et al. (2004, Theorem 1.1) and Rokhlin (2005, Theorem 2) subsequently extended the fundamental theorem to models with random cone constraints on stock holdings.

Napp (2003, Lemma 3.1) removed the assumption of a risk-free numeraire by establishing the fundamental theorem in a discrete-time model with general cone constraints on asset holdings (also on cash), thus removing the need for a risk-free numeraire. It was also Napp (2003, Corollary 4.1) who first established the fundamental theorem of asset pricing in a model with different borrowing and lending rates (and no transaction costs on stock): such a model is free of arbitrage if and only if there exists an artificial interest rate process taking its value between the borrowing and lending rates, and an equivalent probability measure under which the stock price, discounted by this artificial rate, is a martingale.

Pham and Touzi (1999, Application 3.1) observed that in a single-period model with trading at time 0 and time 1, stock holdings can be viewed as two separate assets, one which may only have a non-negative balance, with value equal to the ask price of the stock at time 0 and the bid price at time 1, and another which may only have a non-positive balance with value equal to the bid price of the stock at time 0 and the ask price at time 1. Thus proportional transaction costs can be included into the setting of convex constraints on asset holdings in single-period models. Using this fact, Napp (2003, Corollary 4.2) formulated a version of the fundamental theorem of asset pricing that is very similar to ours in a single-period model with infinite state space.

It is not possible to express proportional transaction costs as convex constraints on asset holdings in multi-period discrete-time models. This is because proportional transaction costs constrain the trades that can be made, rather than the asset holdings themselves. However, in the context of von Neumann–Gale models, Dempster et al. (2006) showed that this framework (of convex cone constraints) can be extended to include proportional transaction costs if, in addition to convex cone constraints on asset holdings, one also requires asset holdings in adjacent periods to jointly satisfy certain convex cone constraints. They established the fundamental theorem of asset pricing in this model, thus simultaneously extending the results of Kabanov and Stricker (2001), Ortu (2001) and Napp (2003).

The model studied in this paper is a special case of the von Neumann–Gale model considered by Dempster et al. (2006). The main result in this paper is obtained by adapting their arguments, which in turn make use of finite-dimensional separation and duality results from linear optimisation, to explicitly take account of different rates for borrowing and lending. For brevity we will use the results of Dempster et al. (2006) directly whenever possible.

Section 1 formally introduces a financial market model with friction in the sense that proportional transaction costs are payable on the stock, and cash accrues interest at different rates of interest, depending on whether the cash balance is positive or negative. Participants in this financial market trade in both assets, subject to reasonable restrictions. We demonstrate in Section 2 that in order to prohibit arbitrage, it is sufficient to be able to embed in our model an artificial friction-free model, consisting of stocks and cash, that is again arbitrage-free. We refer to such an embedded friction-free model as an equivalent martingale triple (see Definition 5). In Section 3 we show that, conversely, if our model admits no arbitrage,

then it must contain an equivalent martingale triple. The main result of this paper is Theorem 8, which establishes the fundamental theorem of asset pricing.

1. The model

Let $(\Omega, \mathcal{F}, \mathbb{Q})$ be a finite probability space with $\mathcal{F} = 2^\Omega$ and $\mathbb{Q}(\omega) > 0$ for all $\omega \in \Omega$. Let $\mathbb{T} := \{0, \dots, T\}$ be the set of trading dates in a discrete-time market model with finite time horizon T . Let $(\mathcal{F}_t)_{t \in \mathbb{T}}$ be a filtration, and denote by \mathcal{L}_t^k the set of \mathbb{R}^k -valued \mathcal{F}_t -measurable random variables at any time $t \in \mathbb{T}$. For convenience we write $\mathcal{F}_{-1} \equiv \mathcal{F}_0$ and $\mathcal{L}_{-1}^k \equiv \mathcal{L}_0^k$.

The model consists of two types of securities, namely a risk-free bond and m risky stocks. Proportional transaction costs apply to trading in shares, i.e. at any time $t \in \mathbb{T}$, stock k can be sold for the bid price $S_t^{kb} \in \mathcal{L}_t^1$ and bought for the ask price $S_t^{ka} \in \mathcal{L}_t^1$. We write $S^b = (S^{1b}, \dots, S^{mb})$ and $S^a = (S^{1a}, \dots, S^{ma})$, and assume that

$$S_t^a \geq S_t^b > 0 \quad \text{for } t \in \mathbb{T}. \tag{1}$$

In this paper all vector inequalities are meant to hold element-wise.

The model also contains a bank account with different lending and borrowing rates, described by predictable interest rate processes r^c (credit rate; for borrowing) and r^d (deposit rate; for lending). For convenience let $r_0^d \equiv r_0^c := 0$. We assume that r^c and r^d satisfy

$$r_t^c \geq r_t^d > -1 \quad \text{for } t \in \mathbb{T}. \tag{2}$$

Define

$$\varrho_t(\alpha) := \alpha^+(1 + r_t^d) - \alpha^-(1 + r_t^c)$$

for $t \in \mathbb{T}$ and $\alpha \in \mathbb{R}$, where $\alpha^+ \equiv \max\{x, 0\}$ and $\alpha^- \equiv \max\{-x, 0\}$. An investment of α units of cash in the bank account at time $(t - 1) \in \mathbb{T} \setminus \{T\}$ accumulates to $\varrho_t(\alpha)$ in cash at time t .

Remark 1. At time 0, a bond with face value 1 expiring at time 1 may be purchased at $1/(1 + r_1^d)$ and sold at $1/(1 + r_1^c) \leq 1/(1 + r_1^d)$. Thus different borrowing and lending rates can be viewed as proportional transaction costs on bonds over single periods.

For convenience, we write $R^d \equiv 1 + r^d$ and $R^c \equiv 1 + r^c$, and define the associated credit and deposit deflator processes by

$$B_t^d := \prod_{s=0}^t (1 + r_s^d) = \prod_{s=0}^t R_s^d, \quad B_t^c := \prod_{s=0}^t (1 + r_s^c) = \prod_{s=0}^t R_s^c$$

for $t \in \mathbb{T}$. Since there is a one-to-one correspondence between descriptions of the bank account in terms of interest rates and deflator process, we adopt the following convention. We reserve B for the deflator process associated with a positive predictable process R , and use these definitions interchangeably, with decorations to indicate the correspondence between individual processes named R and B .

2. Trading and arbitrage

Market agents trade in cash and shares. A market agent is allowed to hold any portfolio (ξ, ζ) consisting of ξ in cash and $\zeta = (\zeta^1, \dots, \zeta^m)$ units of stock at any trading date $t \in \mathbb{T}$: the only (realistic) restrictions are that agents are limited to trades that they can afford, and are not prescient. A market agent seeking to avoid or minimise transaction costs will prefer to aggregate stock trades in order to avoid the simultaneous purchase and sale of stock. Likewise, if the credit interest rate exceeds the deposit rate, then it is not optimal to borrow cash at the same time as having

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