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Some new notions of dependence with applications in optimal allocation problems

Jun Cai [a,*], Wei Wei [b]

[a] Department of Statistics and Actuarial Science, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1

[b] Department of Mathematical Sciences, University of Wisconsin-Milwaukee, Milwaukee, WI, 53211, USA

HIGHLIGHTS

- We propose new dependence notions of RWSAI, COUAI, and UOAI.
- We develop the properties and relationships of these dependence notions.
- We show how these notions of dependence can be constructed by copulas.
- We discuss the applications of these dependence notions in insurance.

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ABSTRACT

Dependence structures of multiple risks play an important role in optimal allocation problems for insurance, quantitative risk management, and finance. However, in many existing studies on these problems, risks or losses are often assumed to be independent or comonotonic or exchangeable. In this paper, we propose several new notions of dependence to model dependent risks and give their characterizations through the probability measures or distributions of the risks or through the expectations of the transformed risks. These characterizations are related to the properties of arrangement increasing functions and the proposed notions of dependence incorporate many typical dependence structures studied in the literature for optimal allocation problems. We also develop the properties of these dependence structures. We illustrate the applications of these notions in the optimal allocation problems of deductibles and policy limits and in capital reserves problems. These applications extend many existing researches to more general dependent risks.

1. Introduction

Optimal allocation problems appear in many fields such as insurance, quantitative risk management, finance, and so on. In the study of insurance, Cheung (2007) considered some interesting questions and models for optimal allocations of deductibles and policy limits. The models and questions of Cheung (2007) have been further generalized and studied in Hua and Cheung (2008), Zhuang et al. (2009), Hu and Wang (2010), Lu and Meng (2011), Li and You (2012), and references therein. The questions and models for optimal allocations of deductibles and policy limits can be formulated as follows.

Let $X_1, \ldots, X_n$ be n losses/risks to be incurred by a policyholder in his n policies and T_i be the occurrence time of loss X_i. The losses vector $(X_1, \ldots, X_n)$ is assumed to be independent of the occurrence times vector $(T_1, \ldots, T_n)$. Through an insurance arrangement of deductibles (policy limits) with an insurer, the policyholder is granted a total deductible (limit) of $d > 0$ over the n policies and the policyholder is allowed to allocate an arbitrary deductible (limit) of d_i with $0 \le d_i \le d$ on risk X_i. If $d_1, \ldots, d_n$ are the allocated deductibles (limits), then $d_i \ge 0$ for all $i = 1, \ldots, n$ and $d_1 + \cdots + d_n = d$. Denote all the admissible allocations of deductibles or limits by D_n, namely, $D_n = \{(d_1, \ldots, d_n) | d_1 + \cdots + d_n = d, d_i \ge 0, i = 1, \ldots, n\}$. Thus, with the insurance arrangements

* Corresponding author.
E-mail address: jcai@uwaterloo.ca (J. Cai).

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of deductibles and policy limits, the total discounted retained loss of the policyholder is $\sum_{i=1}^n e^{-\delta T_i}(X_i \wedge d_i)$ and $\sum_{i=1}^n e^{-\delta T_i}(X_i - d_i)_+$, respectively, where $\delta \geq 0$ is the force of interest. One interesting question is what the optimal deductibles or limits $(d_1^*, \ldots, d_n^*) \in D_n$ are for the policyholder. The policyholder may choose the optimal deductibles or limits $(d_1^*, \ldots, d_n^*)$ to maximize his expected utility of the discounted wealth, namely

$$\max_{(d_1,\ldots,d_n)\in D_n} \mathbb{E}\left[u\left(\omega - \sum_{i=1}^n e^{-\delta T_i}(X_i \wedge d_i)\right)\right] \tag{1.1}$$

or

$$\max_{(d_1,\ldots,d_n)\in D_n} \mathbb{E}\left[u\left(\omega - \sum_{i=1}^n e^{-\delta T_i}(X_i - d_i)_+\right)\right], \tag{1.2}$$

and the policyholder may also choose the optimal deductibles or limits $(d_1^*, \ldots, d_n^*)$ to minimize his expected discounted total retained loss, namely

$$\min_{(d_1,\ldots,d_n)\in D_n} \mathbb{E}\left[\sum_{i=1}^n e^{-\delta T_i}(X_i \wedge d_i)\right] \tag{1.3}$$

or

$$\min_{(d_1,\ldots,d_n)\in D_n} \mathbb{E}\left[\sum_{i=1}^n e^{-\delta T_i}(X_i - d_i)_+\right], \tag{1.4}$$

where ω is the initial wealth of the policyholder after premiums are paid, and u is an increasing and/or concave utility function. Note that if $u(x)$ is increasing and/or concave, then $u^*(x) = -u(\omega - x)$ is increasing and/or convex. Therefore, the optimal allocation problems (1.1)–(1.4) are reduced to the following two types of optimal allocation problems:

$$\min_{(d_1,\ldots,d_n)\in D_n} \mathbb{E}\left[u\left(\sum_{i=1}^n e^{-\delta T_i}(X_i \wedge d_i)\right)\right], \tag{1.5}$$

$$\min_{(d_1,\ldots,d_n)\in D_n} \mathbb{E}\left[u\left(\sum_{i=1}^n e^{-\delta T_i}(X_i - d_i)_+\right)\right], \tag{1.6}$$

where u is an increasing and/or convex function.

In the existing study of the optimal allocation problems (1.5) and (1.6) such as Cheung (2007), Hua and Cheung (2008), Zhuang et al. (2009), Hu and Wang (2010), Lu and Meng (2011), and references therein, the losses $X_1, \ldots, X_n$ were often assumed to have the following independent or comonotonic structures: (i) $X_1, \ldots, X_n$ are mutually independent and $X_1 \leq_{hr} \cdots \leq_{hr} X_n$ or $X_1 \leq_{lr} \cdots \leq_{lr} X_n$, and (ii) $X_1, \ldots, X_n$ are comonotonic and $X_1 \leq_{st} \cdots \leq_{st} X_n$, where the stochastic orders of $\leq_{st}$, $\leq_{hr}$, and $\leq_{lr}$ are defined in Section 2. The similar assumptions on the losses occurrence times $T_1, \ldots, T_n$ were made as well. Recently, Li and You (2012) have studied the optimal allocation problems (1.5) and (1.6) under the same comonotonicity assumptions on the losses $X_1, \ldots, X_n$ as before but a dependence assumption on the losses occurrence times $T_1, \ldots, T_n$. They assumed that $(T_1, \ldots, T_n)$ is linked by a certain Archimedean copula, which implies that the discounted vector $(e^{-\delta T_1}, \ldots, e^{-\delta T_n})$ has an arrangement increasing joint density function, where the definition of an arrangement function will be given in Section 2. These special dependence structures, together with exchangeable losses, are also assumed in other optimal allocation problems such as Cheung and Yang (2004). These restrictions on dependence structures for losses or risks motivate us to consider more general dependence structures.

To determine an optimal allocation, we use the criterion of minimizing the traditional convex risk measure in this paper. For a comprehensive review of other criteria for capital allocations, readers are referred to Dhaene et al. (2012). Recent applications of these allocation principles can be seen in Cheung et al. (2013) and Zaks and Tsanakas (2013).

This paper aims to develop more general dependence structures and to study their applications in optimal allocation problems with dependent risks. The rest of the paper is organized as follows. In Section 2, we present some preliminaries on arrangement increasing functions and stochastic orders. In Section 3, we define the dependence notions of SAI and RWSAI and develop the properties and equivalent characterizations of the two notions. In Section 4, we introduce the dependence notions of UOAI and CUOAI and derive their properties. We conclude that these notions have the implications of SAI $\Longrightarrow$ RWSAI $\Longrightarrow$ CUOAI $\Longrightarrow$ UOAI. In Section 5, we present the properties of marginal distributions of the random vectors with the dependence structures proposed in this paper. We also show how to construct dependent random vectors with these dependence structures through copulas. As applications of these notions of dependence, in Section 6, we consider the optimal allocation problems (1.5) and (1.6) with dependent losses and dependent losses occurrence times. These applications generalize the studies of Cheung (2007), Zhuang et al. (2009), Li and You (2012) to more general dependent risks. Many of their results are special cases of these applications. We also give an application in the allocation problem of capital reserves with RWSAI dependent risks. Section 7 gives some concluding remarks.

2. Preliminaries

In this section, we recall the concept of arrangement increasing functions and the definitions of some stochastic orders, which will be used in this paper.

Throughout the paper, we refer an n-dimensional real-valued vector $(x_1, \ldots, x_n)$ as $\mathbf{x}$ and an n-dimensional random vector $(X_1, \ldots, X_n)$ as $\mathbf{X}$. Accordingly, $\mathbf{X} > (<)\, \mathbf{x}$ means $X_i > (<)\, x_i$ for all $i = 1, \ldots, n$. For any set $K = \{i_1, \ldots, i_k\} \subset \{1, \ldots, n\}$ where $1 \leq i_1 < \cdots < i_k \leq n$ and $k = 1, \ldots, n$, we denote $\mathbf{x}_K = (x_{i_1}, \ldots, x_{i_k})$ and $\mathbf{X}_K = (X_{i_1}, \ldots, X_{i_k})$. For the sake of convenience, we refer the vector $\mathbf{x}$ as $(\mathbf{x}_K, \mathbf{x}_{\bar{K}})$ and $\mathbf{X}$ as $(\mathbf{X}_K, \mathbf{X}_{\bar{K}})$, where $\bar{K} = \{1, \ldots, n\} \setminus K$ is the complement of the set K. In particular, for any $1 \leq i < j \leq n$, if $K = \{i, j\}$, we write $\overline{ij} = \{1, \ldots, n\} \setminus \{i, j\}$, $\mathbf{X}_K = \mathbf{X}_{ij}$, $\mathbf{X}_{\bar{K}} = \mathbf{X}_{\overline{ij}}$, $\mathbf{x}_K = \mathbf{x}_{ij}$, and $\mathbf{x}_{\bar{K}} = \mathbf{x}_{\overline{ij}}$. For example, $\mathbf{X}_{12} = (X_1, X_2)$ and $\mathbf{X}_{\overline{12}} = (X_3, \ldots, X_n)$.

Let $\pi = (\pi(1), \ldots, \pi(n))$ be any permutation of $\{1, \ldots, n\}$, we define $\pi(\mathbf{x}) = (x_{\pi(1)}, \ldots, x_{\pi(n)})$. For any $1 \leq i \neq j \leq n$, we denote the special permutation of transposition by $\pi_{ij} = (\pi_{ij}(1), \ldots, \pi_{ij}(n))$, where $\pi_{ij}(k) = k$ for $k \neq i, j$ and $\pi_{ij}(i) = j$, $\pi(j) = i$.

Furthermore, throughout the paper, 'increasing (decreasing)' means 'non-decreasing (non-increasing)'; all random variables are defined on the common probability space $(\Omega, \mathcal{F}, \mathbb{P})$; expectations under $\mathbb{P}$ are assumed to be finite whenever we write them; the notation of '$\leq_{a.s.} (\geq_{a.s.})$' means the inequality '$\leq$ $(\geq)$' holds almost surely on the common probability space $(\Omega, \mathcal{F}, \mathbb{P})$; and the notation of '$=^d$' means the equality holds in distribution. In addition, we denote the supports of a random variable X and a random vector $\mathbf{X}$ by $S(X)$ and $S(\mathbf{X})$, respectively.

Definition 2.1. A multivariate function $f(\mathbf{x}) = f(x_1, \ldots, x_n)$ is said to be arrangement increasing (AI) if $f(\mathbf{x}) \geq f(\pi_{ij}(\mathbf{x}))$ for any $\mathbf{x} \in \mathbb{R}^n$ and any $1 \leq i < j \leq n$ such that $x_i \leq x_j$. □

Note that a multivariate function $f(\mathbf{x}) = f(x_1, \ldots, x_n)$ is arrangement increasing if and only if $(x_i - x_j)[f(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n) - f(x_1, \ldots, x_j, \ldots, x_i, \ldots, x_n)] \leq 0$ for any $1 \leq i < j \leq n$ and that the arrangement increasing property is preserved