



## Weighted risk capital allocations

Edward Furman<sup>a,\*</sup>, Ričardas Zitikis<sup>b</sup>

<sup>a</sup> Department of Mathematics and Statistics, York University, Toronto, Ontario M3J 1P3, Canada

<sup>b</sup> Department of Statistical and Actuarial Sciences, University of Western Ontario, London, Ontario N6A 5B7, Canada

### ARTICLE INFO

#### Article history:

Received February 2008

Received in revised form

May 2008

Accepted 19 July 2008

#### Keywords:

Weighted risk capital allocation model  
(WRCAM)

Weighted distributions

Weighted premiums

Weighted allocations

Stein's Lemma

General covariance decomposition

Regression function

### ABSTRACT

By extending the notion of weighted premium calculation principles, we introduce weighted risk capital allocations, explore their properties, and develop computational methods. When achieving these goals, we find it particularly fruitful to relate the weighted allocations to general Stein-type covariance decompositions, which are of interest on their own.

© 2008 Elsevier B.V. All rights reserved.

### 1. Introduction and motivation

Let  $X \geq 0$  be a risk random variable (rv) with cumulative distribution function (cdf)  $F_X$ , and let  $\mathcal{X}$  denote the set of all such rv's. Consider a functional  $H : \mathcal{X} \rightarrow [0, \infty]$ . When loaded, that is,  $H[X] \geq E[X]$  for every  $X \in \mathcal{X}$ , then the functional  $H$  is called premium calculation principle (pcp). There is a great variety of pcp's in the literature; see, e.g., Gerber (1979), Bühlmann (1980, 1984), Goovaerts et al. (1984), Denneberg (1994), Kaas et al. (1994), Wang (1995, 1996, 1998), Tsanakas and Desli (2003), Heilpern (2003), Young (2004), Denuit et al. (2005), Denuit et al. (2006), Dhaene et al. (2006), Furman and Landsman (2006) and Furman and Zitikis (2008).

Research in the area of financial risk measurement has fruitfully utilized the concept of actuarial pcp's, and introduced additional challenging and interesting problems. Specifically, consider the pool  $\{X_1, \dots, X_K\}$  of insurance risks  $X_k$ , and denote the overall risk associated with the pool by  $Y$ , which can, for example, be the linear combination  $\sum_{k=1}^K c_k X_k$  or, simply, the sum  $S = \sum_{k=1}^K X_k$ . A non-trivial problem is to quantify the supporting capital, also known as the risk or economic capital, for a contract  $X \in \{X_1, \dots, X_K\}$ .

We denote the capital by  $A[X, Y]$  and refer to the corresponding functional

$$A : \mathcal{X} \times \mathcal{X} \rightarrow [0, \infty]$$

as the risk capital allocation. It should be noted that allocating risk capital is not the final, but rather an intermediate step, which often aims at distinguishing the most profitable business units and thus contributes to the overall efficiency of the decision making process; for details, see, e.g., Tasche (1999), Laeven and Goovaerts (2004), Venter (2004) and Filipović and Kupper (2008).

In this paper (see Section 2) we formulate a class of allocations  $A_w : \mathcal{X} \times \mathcal{X} \rightarrow [0, \infty]$ , which we call weighted risk capital allocations. The allocations are related to the 'weighted' functionals  $H_w : \mathcal{X} \rightarrow [0, \infty]$  defined by (see Furman and Zitikis (2008))

$$H_w[X] = \frac{E[Xw(X)]}{E[w(X)]}, \quad (1.1)$$

where  $w : [0, \infty) \rightarrow [0, \infty)$  is a function, called weight function, which is usually chosen by the decision maker depending on the problem at hand. (All weight functions throughout this paper are deterministic, non-negative, and Borel-measurable.) It is important to note that if the weight function  $w$  is non-decreasing, which is usually the case, then the weighted functional  $H_w$  is loaded and thus defines a rich class of actuarial 'weighted' pcp's. In view of the above, the present paper is therefore a natural continuation of the research by Furman and Zitikis (2008).

\* Corresponding author. Tel.: +1 416 736 2100; fax: +1 416 736 5757.

E-mail addresses: [efurman@mathstat.yorku.ca](mailto:efurman@mathstat.yorku.ca) (E. Furman), [zitikis@stats.uwo.ca](mailto:zitikis@stats.uwo.ca) (R. Zitikis).

The rest of the paper is organized as follows. In Section 2 we formally introduce the weighted allocation  $A_w : \mathcal{X} \times \mathcal{X} \rightarrow [0, \infty]$  and present a number of illustrative examples. In Section 3 we study, discuss, and verify properties of the weighted allocation. In Section 4 we address computational aspects of  $A_w[X, Y]$  for various choices of random pairs  $(X, Y)$  and weight functions  $w$ . As a by-product, we suggest a route, which relies on a modification of Stein’s Lemma, for disentangling the dependence structure between the rv’s  $X$  and  $Y$  from the weight function  $w$ . This, in turn, relates our current research to the capital asset pricing model (CAPM) of Sharpe (1964), Lintner (1965), and Black (1972). Section 5 concludes the paper with an overview of main contributions.

**2. Weighted allocations**

The allocation problem appears in various fields, including operation research, economics, finance. Naturally, the definitions of the phenomenon vary from field to field. In the present paper we define the risk capital allocation induced by a risk measure  $H : \mathcal{X} \rightarrow [0, \infty]$  as a map  $A : \mathcal{X} \times \mathcal{X} \rightarrow [0, \infty]$  such that

$$A[X, X] = H[X] \quad \text{for all } X \in \mathcal{X}. \tag{2.1}$$

We next specialize this definition and introduce the *weighted* allocation, which is the main object of our study in the present paper.

**Definition 2.1.** Let  $X, Y \in \mathcal{X}$ , and let  $w : [0, \infty) \rightarrow [0, \infty)$  be a deterministic Borel function. We call the functional  $A_w : \mathcal{X} \times \mathcal{X} \rightarrow [0, \infty]$ , defined by the equation

$$A_w[X, Y] = \frac{\mathbf{E}[Xw(Y)]}{\mathbf{E}[w(Y)]}, \tag{2.2}$$

the weighted allocation induced by the weighted pcp  $H_w : \mathcal{X} \rightarrow [0, \infty]$ ; see Eq. (1.1).

A number of well known allocation rules are special cases of  $A_w : \mathcal{X} \times \mathcal{X} \rightarrow [0, \infty]$  under appropriate choices of the weight function  $w$ , which can be independent of, or dependent on, the cdf’s  $F_X$  and/or  $F_Y$ ; see Table 2.1. An example of the weighted allocation when  $w$  does not depend on any cdf is Esscher’s allocation, whose detailed study with further references can be found in, e.g., Bühlmann (1980, 1984), Wang (2002, 2007), Goovaerts et al. (2004), Young (2004), Denuit et al. (2005), Pflug and Römisch (2007) and Goovaerts and Laeven (2008). For another example of the weighted allocation when  $w$  is independent of any cdf, we refer to Kamps (1998) where we find a discussion of a pcp that induces Kamps’s allocation (see Table 2.1). For examples of the weighted allocation when  $w$  does depend on  $F_Y$ , we refer to, e.g., Tsanakas and Barnett (2003), Tsanakas (2008) where the distorted allocation is studied (see Table 2.1); to Denault (2001), Panjer and Jing (2001) for the TCE allocations (see Table 2.1); to Furman and Landsman (2006) for the modified tail covariance (MTCov) allocation (see Table 2.1). We note in passing that the TCE allocation  $\mathbf{E}[X|Y \geq F_Y^{-1}(p)]$  is closely related to the absolute concentration curve (ACC)  $p \mapsto \mathbf{E}[X\mathbf{1}\{Y \leq F_Y^{-1}(p)\}]$ , which has played a prominent role in financial portfolio management; see Shalit and Yitzhaki (1994), Schechtman et al. (2008), and references therein.

**Note 2.1.** When comparing or ordering several weighted allocations, we may need to keep in mind that the weight function  $w$  may depend on the cdf  $F_Y$ , in which case we may write  $w_{F_Y}$  or, in a simpler way,  $w_Y$ . In view of this, it is more precise to use the cumbersome notation  $A_{w_Y}[X, Y]$  instead of  $A_w[X, Y]$ , but we prefer the latter one. However, if we feel that this might cause a confusion, then we shall use  $A_{w_Y}[X, Y]$  as we do, for example, in Proposition 3.1.

**Table 2.1**  
Examples of  $A_w[X, Y]$  with various weight functions

Weighted allocations	$w(y)$	Formulas
MCov allocation	$y$	$\mathbf{E}[X] + \mathbf{Cov}[X, Y]/\mathbf{E}[Y]$
Size-biased allocation	$y^t$	$\mathbf{E}[XY^t]/\mathbf{E}[Y^t]$
Esscher’s allocation	$e^{ty}$	$\mathbf{E}[Xe^{ty}]/\mathbf{E}[e^{ty}]$
Kamps’s allocation	$1 - e^{-ty}$	$\mathbf{E}[X(1 - e^{-ty})]/\mathbf{E}[(1 - e^{-ty})]$
Excess-of-loss allocation	$\mathbf{1}\{y \geq t\}$	$\mathbf{E}[X Y \geq t]$
Distorted allocation	$g'(\overline{F}_Y(y))$	$\mathbf{E}[Xg'(\overline{F}_Y(Y))]$
TCE allocation	$\mathbf{1}\{y \geq y_p\}$	$\mathbf{E}[X Y \geq y_p]$
MTCov allocation	$y\mathbf{1}\{y \geq y_p\}$	$\mathbf{E}[X Y \geq y_p] + \mathbf{Cov}[X, Y Y \geq y_p]/\mathbf{E}[Y Y \geq y_p]$

The  $\mathbf{1}\{\cdot\}$  denotes the indicator function and  $y_p = F_Y^{-1}(p)$  the  $p$ -th quantile of the cdf  $F_Y$ . Both  $t \in [0, \infty)$  and  $p \in (0, 1)$  are fixed parameters. Note that the distorted allocation is weighted if  $g$  is differentiable.

In general, the weighted allocation  $A_w[X, Y]$  can be interpreted in several ways. First, it can be viewed as the solution in  $a$  to the minimization problem  $\min_a \mathbf{E}[(X - a)^2 w(Y)]$ . Second,  $A_w[X, Y]$  can be viewed as the mean  $\int \rho(y) dF_{w,Y}(y)$  of the regression function  $\rho(y) = \mathbf{E}[X|Y = y]$  with respect to the weighted cdf

$$F_{w,Y}(y) = \frac{\mathbf{E}[\mathbf{1}\{Y \leq y\}w(Y)]}{\mathbf{E}[w(Y)]}. \tag{2.3}$$

Third, it is useful, and indeed crucial for our following discussion, to observe that the weighted allocation  $A_w[X, Y]$  can be written, and thus interpreted accordingly, as follows:

$$A_w[X, Y] = \mathbf{E}[X] + \frac{\mathbf{Cov}[X, w(Y)]}{\mathbf{E}[w(Y)]}. \tag{2.4}$$

Hence, the ratio  $\mathbf{Cov}[X, w(Y)]/\mathbf{E}[w(Y)]$  can be thought of as a safety loading due to the risk  $X$ . When the random variables  $X$  and  $w(Y)$  are positively correlated, i.e.,  $\mathbf{Cov}[X, w(Y)] \geq 0$ , then we have the bound

$$A_w[X, Y] \geq \mathbf{E}[X], \tag{2.5}$$

which we call the loading property, following the accepted terminology in the context of pcp’s. In particular, bound (2.5) holds when the rv’s  $X$  and  $Y$  are positively quadrant dependent and the weight function  $w$  is non-decreasing (see Lehmann (1966)). We shall discuss further properties of weighted allocations later in this paper. At the moment we only briefly note that while desirable properties of risk measures have been extensively studied in the literature for a long time (see, e.g., Goovaerts et al. (1984), Artzner et al. (1999), Denuit et al. (2005), Dhaene et al. (2006); and references therein), properties of allocations have only relatively recently been postulated and discussed (see, e.g., Denault (2001), Hesselager and Andersson (2002), Valdez and Chernih (2003), Goovaerts et al. (2003) and Dhaene et al. (2008a,b)).

We conclude this section with a note that there are also good reasons to consider a more general functional  $A_{v,w} : \mathcal{X} \times \mathcal{X} \rightarrow [0, \infty]$ , obtained by augmenting the definition of  $A_w$  with a function  $v : [0, \infty) \rightarrow [0, \infty)$  as follows:

$$A_{v,w}[X, Y] = \frac{\mathbf{E}[v(X)w(Y)]}{\mathbf{E}[w(Y)]}.$$

We may think of  $v$  as a utility function, which is usually non-decreasing and concave. Considerations involving conditional tail variance and higher order moments lead to  $v(x) = x^t$  with various  $t \geq 0$ . The allocation  $A_{v,w}[Y, Y]$  with  $v(x) = \mathbf{1}\{x \leq y\}$  is the weighted cdf  $F_{w,Y}(y)$ ; see definition (2.3). In the purely actuarial case, i.e., when  $X = Y$ , the allocation  $A_{v,w}[X, Y]$  reduces (as it should, see property (2.1)) to the generalized weighted premium

$$H_{v,w}[X] = \frac{\mathbf{E}[v(X)w(X)]}{\mathbf{E}[w(X)]},$$

which can be traced back to Heilmann (1989); see also Section 4 in Furman and Zitikis (2008).

متن کامل مقاله

دریافت فوری ←

**ISI**Articles

مرجع مقالات تخصصی ایران

- ✓ امکان دانلود نسخه تمام متن مقالات انگلیسی
- ✓ امکان دانلود نسخه ترجمه شده مقالات
- ✓ پذیرش سفارش ترجمه تخصصی
- ✓ امکان جستجو در آرشیو جامعی از صدها موضوع و هزاران مقاله
- ✓ امکان دانلود رایگان ۲ صفحه اول هر مقاله
- ✓ امکان پرداخت اینترنتی با کلیه کارت های عضو شتاب
- ✓ دانلود فوری مقاله پس از پرداخت آنلاین
- ✓ پشتیبانی کامل خرید با بهره مندی از سیستم هوشمند رهگیری سفارشات