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Risk capital allocation by coherent risk measures based on one-sided moments☆

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Abstract

This paper proposes differentiability properties for positively homogeneous risk measures which ensure that the gradient can be applied for reasonable risk capital allocation on non-trivial portfolios. It is shown that these properties are fulfilled for a wide class of coherent risk measures based on the mean and the one-sided moments of a risky payoff. In contrast to quantile-based risk measures like Value-at-Risk (VaR), this class allows allocation in portfolios of very general distributions, e.g. discrete ones. Two examples show how risk capital given by the VaR can be allocated by adapting risk measures of this class to the VaR.

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1. Introduction

From the works of Denault (2001) and Tasche (2000) it is known that differentiability of risk measures is crucial for risk capital allocation in portfolios. The reason is that in the case of differentiable positively homogeneous risk measures the gradient due to asset weights has figured out to be the unique reasonable per-unit allocation principle. After a short introduction to risk measures at the end of the present section, the approaches of Denault (2001) and Tasche (2000) to this result are briefly reviewed in Section 2 of this paper. However, in contrast to the mentioned result, it is known that in practice quantile-based risk measures like the widely used Value-at-Risk (VaR) methodology or the so-called expected shortfall encounter situations, e.g. in the case of insurance claims, credit portfolios or digital options, where probability distributions are discrete and the risk measures are not differentiable anymore (cf. Tasche, 2000). Furthermore, Section 3 of this paper shows that at least in the case of subadditive positively homogeneous risk measures differentiability on all portfolios actually is not desirable since the risk measures become linear and minimal in this case. As a solution, we define weaker differentiability properties (also

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Section 3). For positively homogeneous (and in particular coherent) risk measures these properties allow allocation by the gradient on all relevant portfolios. Excluded are portfolios that contain only one type of assets. However, in these cases the allocation problem is trivial. In Section 4, we introduce a wide class of coherent risk measures based on the mean and the one-sided moments of a risky payoff. In order to construct the class, it is shown that weighted sums of coherent risk measures are again coherent. Hence, it is possible to "mix" coherent risk measures. For example, one could consider the arithmetic mean of the maximum-loss-principle and a semi-deviation-like risk measure—both are members of the given class. An important result of Section 4 is that the constructed risk measures (expected and maximum loss excluded) are examples for the weakened differentiability properties of Section 3. In contrast to quantile-based risk measures, members of this class allow allocation in portfolios of very general distributions, e.g. discrete ones. Furthermore, for any fixed random payoff X risk measures of this class can be chosen such that the risk capital due to X equals any value between the expected and the maximum loss of X. In Section 5, two numerical examples show how this property can be used to choose a particular risk measure of the class which assigns the same risk capital to a given portfolio as VaR does. As a consequence, the risk capital originally given by the VaR can be allocated by the gradient due to the chosen risk measure. In addition to the mentioned results of the paper, some of the lemmas proven in the technical appendix could be interesting in themselves.

Given a probability space $(\Omega, \mathcal{A}, \mathbb{Q})$, we will consider the vector space $L^p(\Omega, \mathcal{A}, \mathbb{Q})$, or just $L^p(\mathbb{Q})$ for $1 \leq p \leq \infty$. Even though $L^p(\mathbb{Q})$ consists of equivalence classes of p-integrable random variables, we will often treat its elements as random variables. Due to the context, no confusion should arise. The notation will be as follows. We have $\|X\|_p = (\mathbf{E}_{\mathbb{Q}}|X|^p)^{1/p}$ and $\|X\|_\infty = \operatorname{ess\,sup}\{|X|\}$. Recall that $L^p(\mathbb{Q}) \subset L^q(\mathbb{Q})$ if $1 \leq q < p \leq \infty$, since $\|\cdot\|^q \leq \|\cdot\|^p$. X^- is defined as $\max\{-X, 0\}$. We denote $\sigma_p^-(X) = \|(X - \mathbf{E}_{\mathbb{Q}}[X])^-\|_p$. Now, let $U \subset \mathbb{R}^n$ for $n \in \mathbb{N}_+ = \mathbb{N} \setminus \{0\}$ be open and positively homogeneous, i.e. for $u \in U$ we have $\lambda u \in U$ for all $\lambda > 0$. A function $f : U \to \mathbb{R}$ is called positively homogeneous (or homogeneous of degree 1) if $f(\lambda u) = \lambda f(u)$ for all $\lambda > 0$, $u \in U$. When f is also differentiable at every $u = (u_1, \dots, u_n) \in U$, we obtain the well-known Euler theorem

$$f(u) = \sum_{i=1}^{n} u_i \frac{\partial f}{\partial u_i}(u). \tag{1}$$

We consider a one-period framework, that means we have the present time 0 and a future time horizon T. Between 0 and T no trading is possible. We assume "risk" to be given by a random payoff X, i.e. a random variable out of $L^p(\mathbb{Q})$ representing a cashflow at T. We want to consider a risk measure $\rho(X)$ to be the extra minimum cash added to X that makes the position acceptable for the holder or a regulator. For this reason, we state the following definition.

Definition 1.1. A *risk measure* on $L^p(\mathbb{Q})$, $1 \leq p \leq \infty$, is defined by a functional $\rho : L^p(\mathbb{Q}) \to \mathbb{R}$.

We now give a definition of coherent risk measures. For a further motivation and interpretation of this axiomatic approach to risk measurement we refer to the article of Artzner et al. (1999).

Definition 1.2. A functional $\rho : L^p(\mathbb{Q}) \to \mathbb{R}$, where $1 \leq p \leq \infty$, is called a *coherent risk measure* (*CRM*) on $L^p(\mathbb{Q})$ if the following properties hold:

(M) *Monotonicity*: If $X \geq 0$ then $\rho(X) \leq 0$.
(S) *Subadditivity*: $\rho(X + Y) \leq \rho(X) + \rho(Y)$.
(PH) *Positive homogeneity*: For $\lambda \geq 0$ we have $\rho(\lambda X) = \lambda \rho(X)$.
(T) *Translation*: For constants a we have $\rho(a + X) = \rho(X) - a$.

As we work without interest rates—in contrast to Artzner et al. (1999)—there is no discounting factor in Definition 1.2. A generalization of CRM to the space of all random variables on a probability space can be found in Delbaen