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Discrete time hedging with liquidity risk

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ABSTRACT

We study a discrete time hedging and pricing problem in a market with liquidity costs. Using Leland's discrete time replication scheme [Leland, H.E., 1985. Journal of Finance, 1283–1301], we consider a discrete time version of the Black–Scholes model and a delta hedging strategy. We derive a partial differential equation for the option price in the presence of liquidity costs and develop a modified option hedging strategy which depends on the size of the parameter for liquidity risk. We also discuss an analytic method of solving the pricing equation using a series solution.

1. Introduction

Models in mathematical finance theory have been based on the simplified assumption of the competitive and frictionless market. A frictionless market is one without transaction costs and trade restrictions, and a competitive market is a market where traders act as price takers so that their trades do not have any impact on the price process. Although these simplified models are useful first steps for analyzing markets, we need more realistic models without these assumptions to fit the market phenomena better. Recently, models considering market microstructure effects such as liquidity risk, microstructure noises, and information asymmetry have been popular topics.

Liquidity risk is the additional risk in the market due to the timing and size of a trade. The price process may depend on the activities of traders, especially the trading volume. In the past decade or

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two, the literature on liquidity risk has been growing rapidly; for example, see Jarrow (1992, 1994, 2001), Back (1993), Frey (1998), Frey and Stremme (1997), Cvitanic and Ma (1996), Subramanian and Jarrow (2001), Duffie and Ziegler (2003), Bank and Baum (2004), and Çetin et al. (2004).

Among these works, Çetin et al. (2004) developed a rigorous model incorporating liquidity risk into the arbitrage pricing theory. They established a mathematical formulation of liquidity costs, admissible strategies, self-financing strategies, and an approximately complete market. They also showed the two (approximately modified) fundamental theorems of finance hold under the existence of liquidity risk. They also studied an extension of the Black–Scholes economy incorporating liquidity risk as an illustration of the theory.

Built on the asset pricing theory developed in Çetin et al. (2004), we study how the classical hedging strategies should be modified and how the prices of derivatives should be changed in a financial market with liquidity costs, especially when we hedge only at discrete time points. We consider a discrete time version of the Black–Scholes model and a multiplicative supply curve. Using the Leland approximation scheme (Leland, 1985), we obtain a nonlinear partial differential equation which requires the expected hedging error to be zero. We provide an approximate method for solving this equation using a series solution.

The remaining of the paper is organized as follows. Section 2 introduces the Çetin–Jarrow–Protter model, and explains our model in detail. Section 3 derives a nonlinear partial differential equation for the option price which includes liquidity costs. Numerical results are also provided. Section 4 studies an analytic solution for the PDE given in Section 3. Section 5 presents some conclusions.

2. Model

2.1. Background on liquidity costs

This subsection recalls the concepts introduced in the work of Çetin et al. (2004). A basic idea is that a buy-initiated order drives the price up since it removes the best ask prices in the limit order book; similarly, a sell-initiated order drives it down.

We are given a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leqslant t \leqslant T}, \mathbf{P})$ satisfying the usual conditions where T is a fixed time. $\mathbf{P}$ represents the statistical or empirical probability measure. We consider a market with a risky asset (stock) and a money market account. The stock pays no dividend and we assume that the interest rate is zero, without loss of generality. $S(t, x, \omega)$ represents the stock price per share at time $t \in [0, T]$ that the trader pays/receives for an order of size $x \in \mathbf{R}$ given state $\omega \in \Omega$. A positive order ($x > 0$) represents a buy, a negative order ($x < 0$) represents a sale, and the order zero ($x = 0$) corresponds to the marginal trade. For the detailed structure of the supply curve, we refer to Section 2 of Çetin et al. (2004).

A *trading strategy*(portfolio) is a triplet $((X_t, Y_t{:}t \in [0, T]), \tau)$ where X_t represents the trader's aggregate stock holding at time t (units of the stock), Y_t represents the trader's aggregate money market account position at time t (units of money market account), and τ represents the liquidation time of the stock position.[1] Here, X_t and Y_t are predictable and optional processes, respectively, with $X_{0-} \equiv Y_{0-} \equiv 0$.

A *self-financing strategy* is a trading strategy $((X_t, Y_t{:}t \in [0, T]), \tau)$ where X_t is cadlag if $\partial S(t, 0)/\partial x = 0$ for all t,X_t is cadlag with finite quadratic variation ($[X, X]_T < \infty$) otherwise, and

$$Y_t = Y_0 + X_0 S(0, X_0) + \int_0^t X_{u-} dS(u, 0) - X_t S(t, 0) - \sum_{0 \leqslant u \leqslant t} \Delta X_u [S(u, \Delta X_u) - S(u, 0)] - \int_0^t \frac{\partial S}{\partial x} \times (u, 0) d[X, X]_u^c. \tag{2.1}$$

The first line of Eq. (2.1) is the same as the usual self-financing condition without liquidity risk, and the second line of (2.1) represents the loss due to liquidity costs. Therefore, it is natural to define *the liquidity cost* of a self-financing trading strategy (X, Y, τ) by

[1] We do not use the liquidation time τ in this study, and we assume that we have no such obligation. A self financing strategy and the liquidity cost below are still well defined without τ.