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Heterogeneous duopoly with isoelastic demand function

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ABSTRACT

In this paper we analyze a duopolistic market with heterogeneous firms when the demand function is isoelastic (Puu, T., 1991. Chaos in duopoly pricing. Chaos, Solitons and Fractals 1, 573–581.). We consider the same heterogeneous firms of Zhang et al. (Zhang, J., Da, Q., Wang, Y., 2007. Analysis of nonlinear duopoly game with heterogeneous players. Economic Modelling 24, 138–148.) introducing a nonlinearity in the demand function instead of the cost function. Stability conditions of the Nash equilibrium and complex dynamics are studied. In particular we show two different routes to complicated dynamics: a cascade of flip bifurcations leading to periodic cycles (and chaos) and the Neimark-Sacker bifurcation which originates an attractive invariant closed curve. Comparisons with respect to the Puu model and the model of Zhang et al. are performed.

1. Introduction

The main difference between oligopolistic markets and the other market forms is that in the former firms must forecast the behavior of the competitors in order to make the optimal (rational) output choice. It is a strong assumption to consider firms able to perfectly infer the choices of the other firms. It seems more realistic to consider mechanisms through which players form their expectations on the decisions of the competitors. The mathematician Augustin Cournot in his pioneering work (1838) was the first to introduce *naive* expectations in a duopoly market. After that, the case with naive expectations has been deeply analyzed taking in consideration alternative features of the demand and cost functions, (see, for instance, Puu, 1991, 1997, 1998, Agliari et al., 2005, Tramontana et al., 2009).

In more sophisticated models, duopolists are assumed to know their lack of information about the decisions of the opponent and do not immediately jump to the foreseen best reply but they adjust the previous decision towards the new optimum. This kind of mechanism is called *adaptive* and examples of its application to duopolistic markets are given, among the others, in Kopel (1996), Agiza (1999), Bischi and Kopel (2001), Agliari et al. (2005, 2006). In this direction (gradual adjustment towards a better position) moves the recently introduced Local Monopolistic Approximation approach (see Naimzada and Sbragia, 2006 and Bischi et al., 2007).

All the above mentioned models refer to the case of homogeneous players, i.e. both the duopolists adopt the same decision mechanism. There is another branch of the literature in which the focus is in what happens if the firms are heterogeneous. This approach characterizes the works by Leonard and Nishimura (1999), Den-Haan (2001), Agiza et al. (2002), Agiza and Elsadany (2003, 2004). Zhang et al. (2007) analyze a duopoly game with heterogeneous players and nonlinear cost function. In this work we show that removing the nonlinearity from the cost function and introducing a microfounded one in the demand function (Puu, 1991) a higher number of possible scenarios can characterize the dynamics of the output.

The paper is organized as follows. In Section 2 we introduce the model and the nonlinear system describing the dynamics of the productions of the firms. In Section 3 we determine the conditions under which the Nash equilibrium is locally stable. Section 4 is devoted to the analysis of the local bifurcations and to the route to complex dynamics. Numerical simulations are also performed.

2. Model

We consider a market served by two firms producing perfect substitute goods. Let $q_i(t)$, $i=1{,}2$ represent the output of ith duopolist during period t. We assume an isoelastic demand function, founded on the hypothesis of Cobb–Douglas utility function of the agents (Puu, 1991)

$$p=\frac{1}{Q}=\frac{1}{q_1+q_2} \tag{1}$$

where Q is the total supply. The cost function is linear

$$C_i(q_i)=c_iq_i,\ \ i=1,2 \tag{2}$$

so the marginal cost is constant and equal to $c_i>0$.

As a consequence of the assumptions concerning demand function and cost function, the profit of the ith firm is given by

$$\Pi_i(q_i,q_{-i})=\frac{q_i}{q_i+q_{-i}}-c_iq_i \tag{3}$$

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where the term q_{-i} is the output of the other duopolist and its presence in the profit function is typical of oligopolistic markets.The marginal profit of the ith duopolist is

$$\phi_i(q_i, q_{-i}) = \frac{\partial \Pi_i(q_i, q_{-i})}{\partial q_i} = \frac{q_{-i}}{(q_i + q_{-i})^2} - c_i \tag{4}$$

Firms do not know in advance the output of the concurrent duopolist so they are not able to compute the output that maximizes their profit. We consider heterogeneous firms in the sense that they adopt different mechanisms to decide the output of each time period. In particular, following Zhang et al. (2007), our first firm (*boundedly rational* firm henceforth) adopts the so-called *myopic* adjustment mechanism (Dixit, 1986)

$$q_1(t+1) = q_1(t) + \alpha_1 q_1(t) \phi_1(q_1(t), q_2(t)) \tag{5}$$

The economic intuition behind the mechanism (5) is that the boundedly rational firm increases/decreases its output according to the information given by the marginal profit of the last period. The positive parameter α_1 represents the speed of adjustment.

The second duopolist, instead, is a *naive* player (*naive* firm henceforth) which expects a production of the rival equal to the last period's one, that is $q^e_{-i}(t+1) = q_{-i}(t)$. Given this assumption concerning the expectations the naive firm will choose the level of output that maximizes its expected profit, i.e. such that

$$\phi_2(q_2(t+1), q_1(t)) = 0 \tag{6}$$

that is

$$q_2(t+1) = \sqrt{\frac{q_1(t)}{c_2}} - q_1(t) \tag{7}$$

The dynamics of the firms' output are expressed by the following discrete time dynamical system

$$T(q_1, q_2) : \begin{cases} q_1' = q_1 + \alpha_1 q_1 \left[\dfrac{q_2}{(q_1 + q_2)^2} - c_1 \right] \\ q_2' = \sqrt{\dfrac{q_1}{c_2}} - q_1 \end{cases} \tag{8}$$

where $'$ denotes the unit-time advancement operator, and we are interested only in positive trajectories. Note also the map is not defined in the origin $(0,0)$.

3. Nash equilibrium

Setting the fixed point conditions $q_1' = q_1 = q_1^*$ and $q_2' = q_2 = q_2^*$ in the map (8) we obtain the following system

$$\begin{cases} \alpha_1 q_1^* \left[\dfrac{q_2^*}{(q_1^* + q_2^*)^2} - c_1 \right] = 0 \\ \sqrt{\dfrac{q_1^*}{c_2}} - q_1^* - q_2^* = 0 \end{cases} \tag{9}$$

which is only solved by the point:

$$E = \left(\frac{c_2}{(c_1 + c_2)^2}, \frac{c_1}{(c_1 + c_2)^2} \right) \tag{10}$$

The unique fixed point E is also the Nash equilibrium of the game, which is the same of the Puu (1991). The coordinates of E imply that in equilibrium the output produced by the more efficient firm is higher than the rival's one. The profits corresponding to the Nash equilibrium are

$$\Pi_1^* = c_2^2 (c_1 + c_2)^2, \quad \Pi_2^* = c_1^2 (c_1 + c_2)^2 \tag{11}$$

and, again, the more efficient firm achieves a higher profit. Hence, in the equilibrium, in order to achieve a higher profit, it is not relevant the mechanism adopted to adjust the output but the efficiency of the firm.

In order to investigate the local stability of the Nash equilibrium we need the Jacobian matrix of the system (8):

$$J(q_1, q_2) : \begin{bmatrix} 1 - \alpha_1 c_1 + \alpha_1 \left[\dfrac{q_2^2 - q_1 q_2}{(q_1 + q_2)^3} \right] & \alpha_1 \left[\dfrac{q_1^2 - q_1 q_2}{(q_1 + q_2)^3} \right] \\ \dfrac{1}{2\sqrt{q_1 c_2}} - 1 & 0 \end{bmatrix} \tag{12}$$

which evaluated at E becomes

$$J(E) : \begin{bmatrix} 1 - \dfrac{2\alpha_1 c_1 c_2}{c_1 + c_2} & \alpha_1 c_2 \left(\dfrac{c_2 - c_1}{c_1 + c_2} \right) \\ \dfrac{c_1 - c_2}{2 c_2} & 0 \end{bmatrix} \tag{13}$$

and its characteristic polynomial can be written as

$$p(\lambda) = \lambda^2 - Tr(J)\lambda + Det(J) \tag{14}$$

where $Tr(J)$ and $Det(J)$ are the trace and the determinant of the Jacobian matrix (13), given by

$$Tr(J) = 1 - \frac{2\alpha_1 c_1 c_2}{c_1 + c_2} \quad ; \quad Det(J) = \frac{\alpha_1 (c_2 - c_1)^2}{2(c_1 + c_2)} \tag{15}$$

The stability conditions are

$$\begin{aligned} &(i)\ p(1) = 1 - Tr(J) + Det(J) > 0 \\ &(ii)\ p(-1) = 1 + Tr(J) + Det(J) > 0 \\ &(iii)\ 1 - Det(J) > 0 \end{aligned} \tag{16}$$

Condition (i) is always fulfilled because

$$p(1) = \frac{2\alpha_1 c_1 c_2}{c_1 + c_2} + \frac{\alpha_1 (c_2 - c_1)^2}{2(c_1 + c_2)} > 0 \quad \forall \alpha_1, c_1, c_2 > 0 \tag{17}$$

whereas conditions (ii) and (iii) define surfaces in the parameter space on which a *Flip* bifurcation and a *Neimark-Sacker* bifurcation take place, respectively.

3.1. Local stability analysis

If conditions (16) are all fulfilled then the Nash equilibrium is locally asymptotically stable. In particular, condition (ii) is satisfied $\forall \alpha_1 > 0$ if $4c_1 c_2 - (c_2 - c_1)^2 \leq 0$ while in the case $(4c_1 c_2 - (c_2 - c_1)^2 > 0)$ (ii) is fulfilled iff $\alpha_1 < \alpha_1^f$ where

$$\alpha_1^f = \frac{4(c_1 + c_2)}{4c_1 c_2 - (c_2 - c_1)^2} \tag{18}$$

Condition (iii) implies that $\alpha_1 < \alpha_1^{NS}$ with

$$\alpha_1^{NS} = \frac{2(c_1 + c_2)}{(c_2 - c_1)^2} \tag{19}$$