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Concept lattices of fuzzy contexts: Formal concept analysis vs. rough set theory

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ABSTRACT

This paper presents a comparative study of concept lattices of fuzzy contexts based on formal concept analysis and rough set theory. It is known that every complete fuzzy lattice can be represented as the concept lattice of a fuzzy context based on formal concept analysis [R. Bělohlávek, Concept lattices and order in fuzzy logic, Ann. Pure Appl. Logic 128 (2004) 277–298]. This paper shows that every complete fuzzy lattice can be represented as the concept lattice of a fuzzy context based on rough set theory if and only if the residuated lattice $(L, *, 1)$ satisfies the law of double negation. Thus, the expressive power of concept lattices based on rough set theory is weaker than that of concept lattices based on formal concept analysis.

1. Introduction

Both the theory of formal concept analysis (FCA) [9] and that of rough set (RST) [10,11,19,20,25,26,28] are useful tools for qualitative data analysis. Formal contexts provide a common framework for both theories. A formal context is a triple (X, Y, R), where X, Y are sets, $R \subseteq X \times Y$ is a relation from X to Y. In a formal context (X, Y, R), X is interpreted as the set of *objects*, Y the set of *properties*, and $(x, y) \in R$ reads as that the object x has property y. Given a context (X, R), there are two Galois connections between the powersets of X and Y [8]. One is the contravariant pair $(R_\uparrow, R^\downarrow)$, which plays a fundamental role in formal concept analysis; the other is the covariant pair $(R_\exists, R^\forall)$, which plays a key role in rough set theory.

A formal concept of a context (X, Y, R) (or, a concept of (X, Y, R) based on formal concept analysis) is a pair $(U, V) \in 2^X \times 2^Y$ such that $U = R^\downarrow(V)$ and $V = R_\uparrow(U)$. The formal concepts of a context (X, Y, R) form a complete lattice in a natural way, called the concept lattice of (X, Y, R) (based on formal concept analysis). The Fundamental Theorem of formal concept analysis asserts that every complete lattice can be represented as the concept lattice of some formal context [9].

In 2002, Düntsch and Gediga [10] introduced the notion of property oriented concepts (or, concepts based on rough set theory) making use of the covariant Galois connection $(R_\exists, R^\forall)$ [8] instead of the contravariant $(R_\uparrow, R^\downarrow)$. The set of the property oriented concepts of a context (X, Y, R) is a complete lattice, called the property oriented concept lattice (or, the concept lattice based on rough set theory). Each complete lattice can also be represented as the concept lattice of some formal context (X, Y, R) based on rough set theory [27]. Therefore, the concept lattices of formal contexts based on rough set theory have the same expressive power as the concept lattices of formal contexts based on formal concept analysis.

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Both the theories of formal concept analysis and rough set theory have been generalized to the fuzzy setting [3–5,11]. Let $(L,*,1)$ be a complete residuated lattice. A fuzzy (formal) context is a triple (X,Y,R), where $R : X \times Y \to L$ is a fuzzy relation between the sets X and Y. For a fuzzy context (X,Y,R), Bělohlávek [1] introduced a contravariant Galois connection $(R_\uparrow, R^\downarrow)$ between the fuzzy powersets L^X and L^Y. Making use of the Galois connection $(R_\uparrow, R^\downarrow)$, Bělohlávek [5] introduced the concept of a formal concept of the fuzzy context (X,Y,R). The Fundamental Theorem of formal concept analysis has been extend to the fuzzy situation in [5]. Precisely, Bělohlávek introduced the notion of complete L-ordered sets (or, complete L-lattices for short), which is in fact a notion of complete lattices in fuzzy logic, then he proved that for any fuzzy context (X,Y,R), the set $\mathfrak{B}(X,Y,R)$ of all concepts of (X,Y,R) is a complete L-lattice; conversely, every complete L-lattice is isomorphic to $\mathfrak{B}(X,Y,R)$ for some fuzzy context (X,Y,R).

For a fuzzy context (X,Y,R), a covariant Galois connection $(R_\exists, R^\forall)$ between the fuzzy powersets L^X and L^Y has been defined in [11,21]. This covariant Galois connection is a fundamental tool in the study of (generalized) fuzzy rough set theory. Analogous to the classical situation, the concept of a property oriented concept of a fuzzy context is introduced in terms of $(R_\exists, R^\forall)$ [10,20,26]. The set of property oriented concepts of (X,Y,R) is denoted by $\mathfrak{P}(X,Y,R)$. As we shall see in the sequel, for any fuzzy context (X,Y,R), $\mathfrak{P}(X,Y,R)$ is also a complete L-lattice. So, a natural question is:

Question 1.1 Whether every complete L-lattice is isomorphic;to $\mathfrak{P}(X,Y,R)$ for some fuzzy context (X,Y,R)?

The answer to this question is, a little surprisingly, negative in general. Precisely, it is shown that

(i) a complete L-lattice is isomorphic to the concept lattice of some fuzzy context based on rough set theory if and only if it is isomorphic to a fuzzy opening system in some fuzzy powerset L^X;
(ii) every complete L-lattice is isomorphic to a fuzzy opening system in some fuzzy powerset L^X if and only if $(L,*,1)$ satisfies the law of double negation.

Therefore, if $(L,*,1)$ does not satisfy the law of double negation, then there exists a fuzzy complete lattice that is not isomorphic to the concept lattice of any fuzzy context based on rough set theory. Thus, the expressive power of concept lattices of fuzzy contexts based on formal concept analysis is, in general, stronger than that based on rough set theory.

In order to make clear the connection and difference between concept lattices of fuzzy contexts based on formal concept analysis and rough set theory, a comparative study of the two theories is undertaken in this paper. As by-products, some new characterizations of formal concept lattices of fuzzy contexts are also obtained.

The contents are arranged as follows. Section 2 presents a brief introduction of concept lattices based on formal concept analysis and rough set theory. Section 3 recalls some basic notions of L-ordered sets and complete L-lattices needed in the sequel. Section 4 focuses on concept lattices of fuzzy contexts based on formal concept analysis. Section 5 is devoted to concept lattices of fuzzy contexts based on rough set theory.

2. Concept lattices based on formal concept analysis and rough set theory

For convenience of the reader, we recall in this section some basic facts about concept lattices based on the formal concept analysis and that based on rough set theory.

Given a context (X,Y,R), define a pair of operators $(R_\uparrow, R^\downarrow)$ between the powersets of X and Y as follows:

$$R_\uparrow : 2^X \to 2^Y, \quad R_\uparrow(U) = \{y \in Y : \forall x \in U, xRy\}; \tag{1}$$

$$R^\downarrow : 2^Y \to 2^X, \quad R^\downarrow(V) = \{x \in X : \forall y \in V, xRy\}. \tag{2}$$

This pair of operators $(R_\uparrow, R^\downarrow)$ is a contravariant Galois connection between the powersets of X and Y. A *formal concept* [9] of a context (X,Y,R) (or, a concept of (X,Y,R) based on formal concept analysis) is a pair $(U,V) \in 2^X \times 2^Y$ such that $U = R^\downarrow(V)$ and $V = R_\uparrow(U)$. U is called the *extent* and V is called the *intent*. The set of all the formal concepts of a context (X,Y,R) is denoted by $\mathfrak{B}(X,Y,R)$.

Given two concepts $(U_1,V_1),(U_2,V_2)$ of a context (X,Y,R), it is easily seen that $U_1 \subseteq U_2 \iff V_2 \subseteq V_1$. Define a partial order on the set of all the formal concepts of a context (X,Y,R) as follows:

$$(U_1,V_1) \leqslant (U_2,V_2) \iff U_1 \subseteq U_2 \iff V_2 \subseteq V_1.$$

Then the set $\mathfrak{B}(X,Y,R)$ equipped with the order $\leqslant$ is a complete lattice. In fact, given a family $\mathscr{U} = \{(U_i,V_i), i \in I\}$ of formal concepts of (X,Y,R), it holds that

$$\bigvee \mathscr{U} = \left(R^\downarrow\left(\bigcap_{i\in I} V_i\right), \bigcap_{i\in I} V_i\right), \quad \bigwedge \mathscr{U} = \left(\bigcap_{i\in I} U_i, R_\uparrow\left(\bigcap_{i\in I} U_i\right)\right).$$

The following theorem is called the Fundamental Theorem of concept lattices in [7].

Theorem 2.1. *Let V be a complete lattice and (X,Y,R) a context. Then V is isomorphic to $\mathfrak{B}(X,Y,R)$ if and only if there exist mappings $\gamma : X \to V$ and $\delta : Y \to V$ such that $\gamma(X)$ is $\bigvee$-dense in V, $\delta(Y)$ is $\bigwedge$-dense in V, and $(x,y) \in R \iff \gamma(x) \leqslant \delta(y)$ for all $x \in X$ and $y \in Y$.*