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The influence of bankruptcy value on optimal risk control for diffusion models with proportional reinsurance

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Abstract

This paper considers the model of a financial entity such as an insurance company whose surplus is governed by a Brownian motion with constant drift and diffusion coefficient. A proportional reinsurance available to the company allows it to reduce its risk by simultaneously reducing the diffusion coefficient and the drift. The uncontrolled dividends are accumulated at the rate proportional to the current value of the surplus. It is assume that at the time of bankruptcy the company liquidation (*bankruptcy* or *terminal*) value is P. The objective is to find the policy which maximizes the total discounted value of dividends and the terminal value of the company. We find the optimal policy and analyze its dependence on P.

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1. Introduction

In the optimization models in mathematical finance the issues of bankruptcy loom large especially in the consumption/investment models, where exercising one of the controls results in the decrease of wealth. There were quite a few papers dealing with the bankruptcy issue in the models of optimal consumption/investment behavior of a small investor, in particular the dependence of the optimal consumption/investment policy on the terminal (bankruptcy) value. One can find a rather thorough treatment as well as a comprehensive list of references in the book by Sethi (1997).

On the other hand the same problem in the context of the dividend/risk control optimization, which often appears when one deals with the optimization issues in the context of the insurance entity, has not been seriously investigated so far. There have been several papers which study optimal dynamical control of reinsurance rates and/or dividends

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(e.g., Asmussen et al. (2000), Højgaard and Taksar (1998a,b) and Taksar (2000b)). However in the context of the insurance models the problem of dependence of the optimal policy on the terminal value at bankruptcy has not been studied in earnest. There is only one paper which deals with this problem for the diffusion model when one controls both the risk and the dividend payouts (see Taksar (2000a)).

Surprisingly when the payouts are not controlled the problem becomes significantly more difficult. In this paper we investigate the model in which the dividends are paid-off at the rate proportional to the current surplus, while only the risk in the form of reinsurance is controlled. While the problem of fixed rate dividends stems from a different economic perspective than that of the controllable dividends, it is interesting in its own right and the optimal policy in this model yields rather interesting insights.

Usually the insurance company – called *cedent* – is risk averse and does not want full risk exposure for all the incoming claims. The means to reduce the risk is reinsurance. Reinsurance is a contract when the cedent is paying a part of the incoming premium to another insurance company, called a *reinsurance company*, in exchange for an obligation from the latter to pick up some fraction of the risk, whereby the cedent reduces its own risk as well as potential profit. In this paper we will consider the proportional reinsurance, which corresponds to the reinsurance company paying a fixed fraction of each claim, simultaneously receiving the same fraction of the premiums from the cedent. This fraction is dynamically controlled by the cedent whose objective is to maximize the total discounted expected payoff until the time of bankruptcy.

By the term *bankruptcy* we refer to the state of zero surplus. This is the case when the company's liquid assets vanish. Various assumptions about what happens at the time of bankruptcy and after can be made. One can however associate a financial value to the bankruptcy or the economic activity after that (in the models of bankruptcy with recovery). We incorporate all such models by assigning a value P to bankruptcy, and including P as a parameter of our model. When the value P is negative, the company is fined for going into bankruptcy. When P is positive, we can interpret it as the value that has accrued from the sale of non-liquid assets.

In the next section we rigorously formulate the mathematical model and describe the stochastic control problem associated with it. We also set the corresponding Hamilton–Jacobi–Bellman (HJB) equation which the *optimal value* or the *optimal return* function of the problem should satisfy.

In Section 3 we solve the corresponding HJB equation and find the value function of the problem. The last section is devoted to the economic analysis of the obtained results.

2. Stochastic control model

In the case when there is no diffusion approximation the *risk process* or the *surplus* representing the liquid assets of an insurance company, is modeled by a compound Poisson process (the so called classical *Cramer–Lundberg model*)

$$R(t) = x + pt - \sum_{i=1}^{N_t} U_i.$$

Here x is the company's initial capital, p is the premium rate, $\{N_t\}_{t\geq 0}$ is a Poisson process with the intensity β, whose value corresponds to the number of claims occurring in the time interval $[0, t]$, and U_i is the size of the i-th claim. The claims are i.i.d. with the first and the second moments m and s^2 respectively. In the case of *proportional* reinsurance, the company reinsures a fraction of the claims by setting the retention level $a \in [0, 1]$. In this case the cedent covers the fraction a of each claim, and the reinsurance company covers the rest, namely the liability of the cedent for the payments on the i-th claim is reduced from U_i to aU_i while the premium rate is also reduced from p to ap. Thus the corresponding surplus process becomes

$$R(t) = x + apt - a\sum_{i=1}^{N_t} U_i. \tag{2.1}$$

This process can be approximated by a Brownian motion with the drift $a\mu$ and diffusion $a\sigma$ where $\mu = p - \beta m$ and $\sigma^2 = \beta s^2$. In this paper we consider the diffusion approximation for the surplus process with the retention level (i.e., our proportionality factor being dynamically controlled).

In a rigorous mathematical formulation of the problem we start with a probability space $(\Omega, \mathcal{F}, \mathcal{P})$, endowed with the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ and a standard Brownian motion $w(t)$ adapted to $\mathcal{F}_t$.