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Existence of equilibrium in a differential game of spatial competition with advertising

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ABSTRACT

We investigate a differential duopoly game with horizontal product differentiation and advertising efforts aimed at increasing market demand, to show that the standard approach to spatial competition fails to produce a pure-strategy price equilibrium in a dynamic game framework. This holds independently of the shape of the transportation cost function. Then, we introduce an endogenous cost associated with the choice of location and characterise the feedback equilibrium, identifying the necessary and sufficient condition for the existence of the pure-strategy (stationary) price equilibrium. The same condition is singled out for the static game where consumer population is constant. Finally, we show that the static game cannot be viewed as a special case of the dynamic one.

1. Introduction

We propose a dynamic approach to the strategic use of non-price tools in a differential game model of spatial competition. Non-price variables typically include product and/or process R&D, product differentiation and advertising, that firms may use in isolation or together, so as to increase the profitability of their price or quantity strategies. Here, we focus on (i) horizontal differentiation, and (ii) advertising investments aimed at increasing demand (or market size).

Ever since Hotelling's (1929) seminal contribution, the role of product differentiation as a remedy to the fragility of market equilibrium under price competition has represented a core issue in the field of industrial organization.

However, under horizontal product differentiation, an established result is that a pure-strategy equilibrium in prices may not always exist.[2] More precisely, a subgame perfect equilibrium with prices greater than marginal cost may fail to exist, because of an undercutting incentive operating when transportation costs are linear (or not sufficiently convex) in the distance between the generic consumer and the firm he decides to patronise. This non-existence problem has generated a stream of literature proposing several remedies, either by adopting non-linear transportation cost functions (d'Aspremont et al., 1979; Stahl, 1982; Economides, 1986) or by adopting the Stackelberg equilibrium as the solution concept (Anderson, 1987), or by choosing the appropriate distribution functions for the population of consumers (de Palma et al., 1985; Neven, 1986), or a mix thereof (Tabuchi and Thisse, 1995; Lambertini, 1997a).

These remedies work in 'location-then-price' games, i.e., if the game is solved by backward induction with different variables being set at different stages. Novshek (1980) establishes that, if firms choose prices and locations simultaneously, then a pure-strategy Nash equilibrium fails to exist due to a well-known undercutting argument. This holds independently of consumer distributions and transportation cost functions, the only condition being that marginal costs must not be too steep. However, the usual procedure implementing the backward induction algorithm stage by stage, usually adopted in static multistage games of product differentiation, cannot be used to solve the continuous-time differential game formulations of the same problems.[3]

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[2] For exhaustive accounts of the debate, see Caplin and Nalebuff (1991); Anderson et al. (1992); Anderson et al. (1997).

[3] Indeed, all subgame perfect equilibria of differential games rely on a backward induction argument. However, this is not employed in the same way as in static games, since in continuous-time problems all first order conditions are taken simultaneously.

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We focus on this problem using as a benchmark the linear transportation cost as in Hotelling (1929), assuming that firms invest in advertising in order to increase the population of consumers (as in Piga, 1998).[4] That is, advertising is modelled as a public good. First we briefly outline the non-existence problem, and then we modify the setup to allow for a cost associated with adjusting locations. We establish the necessary and sufficient condition ensuring the existence of a price equilibrium in pure strategies *at all times during the game* and we characterise the steady state equilibrium of the system under feedback information. Then, we also sketch the equilibrium of the static two-stage game where the population is constant, and we compare it to the outcome of the differential game *for a given population size*, to find that (i) the degree of differentiation is larger in the static game than in the dynamic one, and consequently (ii) profits are higher in the former than in the latter. This shows that applying the backward induction in the two settings yields two largely different pictures, with the static pure-strategy equilibrium being sustainable in a wider parameter range.

The remainder of the paper is structured as follows. Section 2 illustrates the basic setup and the non-existence issue. Section 3 is devoted to the analysis of the differential game, while the outline of the static game and the comparative assessment of both are in Section 4. Concluding remarks are in Section 5.

2. The setup

We consider a market for horizontally differentiated products *à la* Hotelling (1929). The market exists over $t\in[0,\infty]$. Two profit-maximising firms, 1 and 2, choose locations $x_1(t)$ and $x_2(t)\in[0,1]$ and compete in prices simultaneously. Unit production cost $c>0$ is assumed to be constant and equal across firms. Throughout the time horizon considered, firms have the same discount rate $\rho\in[0,1]$.

Consumers are uniformly distributed with density $N(t)$ along the unit interval [0,1]. At any t, the total mass of consumers is therefore $N(t)$. The generic consumer located at $m\in[0,1]$ buys one unit of the good, enjoying the following net surplus:

$$U = s - p_i(t) - g(x_i(t) - m) \geq 0, \qquad i = 1, 2, \tag{1}$$

where x_i and p_i are firm's i location and mill price, respectively; $g(x_i - m)$ is the transportation cost function. In the remainder of the paper, we suppose that the reservation price s is never binding, so that full market coverage always obtains. If

$$g(x_i - m) \equiv k|x_i - m|, \tag{2}$$

the model keeps Hotelling's original assumption of linear disutility of transportation. Therefore, the consumer indifferent between products 1 and 2 is located at[5]:

$$\overline{m}(t) = \frac{p_2(t) - p_1(t) + k(x_1(t) + x_2(t))}{2k}, \tag{3}$$

and the associated demands are:

$$y_1(t) = N(t)\ \overline{m}(t) = \frac{N(t)[p_2(t) - p_1(t) + k(x_1(t) + x_2(t))]}{2k}; \qquad y_2(t) = N(t) - y_1(t). \tag{4}$$

Firms can increase the level of demand over time through the following dynamic equation *à la* Nerlove and Arrow (1962):

$$\dot{N}(t) \equiv \frac{dN(t)}{dt} = \alpha[A_1(t) + A_2(t)] - \delta N(t),\ \alpha > 0, \tag{5}$$

where $A_i(t)$ is the advertising effort carried out by firm i at time t, $\delta\in[0,1]$ is the constant decay rate of demand, and α measures the marginal impact of advertising on demand.[6] This type of advertising is a pure public good as the effort carried out by any firm benefits all firms alike (see Fershtman, 1984; Fershtman and Nitzan, 1991); accordingly, it is sometimes referred to as *cooperative*, with the implicit *caveat* that firms do not cooperate in the sense of joint profit maximisation.[7] The instantaneous cost of advertising for firm i is[8]:

$$C_i(A_i(t)) = b[A_i(t)]^2, b > 0. \tag{6}$$

The dynamics of consumer population (Eq. (5)) generated through advertising is not only interesting *per se*, but also (and perhaps more) because it is unaffected by prices and locations. I.e., it is unrelated to the control variables directly associated with the non-existence problem.

Firm i's instantaneous profits are:

$$\pi_i(t) = [p_i(t) - c_i]y_i(t) - b[A_i(t)]^2, \tag{7}$$

where $y_i(t)$ is given by Eq. (4). Firm i's Bellman equation is:

$$\rho V_i(N(t)) = \max_{p_{i(t)},x_i(t)A_i(t)} \left\{ [p_i(t) - c_i]y_i(t) - b[A_i(t)]^2 + \frac{\partial V_i(N(t))}{\partial N(t)}[\alpha(A_1(t) + A_2(t)) - \delta N(t)] \right\} \tag{8}$$

where the control variables are $\{p_i(t),x_i(t),A_i(t)\}$, the state variable (common to both firms) is $N(t)$, and $V_i(N(t))$ is firm i's value function associated to $N(t)$.

Piga (1998) shows the coincidence between the open-loop equilibrium and the feedback equilibrium if $x_1 = 0$ and $x_2 = 1$. Therefore, at least for these locations, the feedback equilibrium does exist in pure strategies and is also subgame perfect. Once we allow firms to choose locations, it is easily shown that a pure-strategy equilibrium does not exist. This result can be proved without developing the related calculations explicitly.

Indeed, this game cannot produce a pure-strategy equilibrium in prices, *irrespective* of the shape of transportation costs. As we know from Novshek (1980), if firms choose prices and locations simultaneously, then a pure-strategy Nash equilibrium fails to exist. In particular, (i) there can exist no equilibrium with firms located at different points, because then a firm would profit by choosing a location close to (or the same as) the rival's and undercut her price;

[4] The relationship between advertising and the degree of differentiation depends upon the nature of advertising one has in mind. Although a complete overview of this theme is outside the scope of the present paper, several contributions are to be mentioned. For persuasive advertising, see von der Fehr and Stevik (1998). Informative advertising has been extensively investigated: see Grossman and Shapiro (1984), Meurer and Stahl (1994), Bester and Petrakis (1995) and Vettas (1998), *inter alia*.

[5] We omit the indifference condition as well as the derivation of the expression for $\overline{m}(t)$, as they are well-known from previous literature (see d'Aspremont et al., 1979, *inter alia*).

[6] Throughout the paper, for the sake of simplicity we assume that α is the same for both firms. However, it can be argued that the marginal productivity of advertising may not be symmetric. In such a case, given the public good nature of advertising, the firm characterised by a higher α can be expected to invest less than the rival. On this, as well as for an exhaustive overview of the large literature on dynamic models of advertising, see Feichtinger et al. (1994); Dockner et al. (2000, ch. 11); and Jørgensen and Zaccour (2004), *inter alia*.

[7] This labelling dates back to Friedman (1983).

[8] According to the cost function in Eq. (6), advertising exhibits decreasing returns to scale. On the empirical evidence supporting this assumption, see Feichtinger et al. (1994).