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A non-linear dynamic model of the variance risk premium

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ABSTRACT

We propose a new class of non-linear diffusion processes for modeling financial markets data. Our non-linear diffusions are obtained as transformations of affine processes. We show that asset-pricing and estimation is possible and likelihood estimation is straightforward. We estimate a non-linear diffusion model for the VIX index under both the objective measure and the risk-neutral measure where the latter is obtained from futures prices. We find evidence of significant non-linearity under both measures. We define the difference between the P and Q drift as a measure of the variance risk premium and show that it has strong predictive power for stock returns.

1. Introduction

Non-linear diffusion processes are appealing tools for modeling financial prices. Non-linearities in the drift of a diffusion can imply that a process will mean-revert quickly from high levels. For example, Aït-Sahalia (1996), Stanton (1997), and Bandi (2002) among others, find that interest rates exhibit such non-linearities. Bandi and Reno (2012) study non-parametric jump diffusions for volatility. There is also considerable evidence that many economic processes do not exhibit exponentially decaying autocorrelation functions, as implied by linear models.

Despite the appealing characteristics of non-linear models in representing economic data such as nominal interest raters, and volatility, their use in pricing applications is limited. The primary reason for this is that general, unrestricted non-linear models, while providing excellent fit to data, are typically difficult to use in asset pricing applications as asset prices are only available through computationally intensive methods. Moreover, unrestricted non-linear processes are difficult to constrain to a subspace of $\mathbb{R}^n$ which is sometimes required (i.e for positivity and or stationarity).

This paper makes two contributions. First, we propose a method for deriving a class of non-linear diffusion processes that have the desirable properties that allow us to use them in modeling financial market prices. Our class of such diffusion processes is derived as non-linear transformations of basic affine processes. The originating process, say x_t can be an OU or CIR process. The transformed process, $y_t = f(x_t)$, has the following three desirable properties

1. It can be written as a diffusion
$$dy_t = \mu_y(y_t)dt + \sigma_y(y_t)dB_t \quad (1)$$
where μ_y and σ_y are non-linear drift and diffusion functions.
2. The transition density $p(y_{t+s} \mid y_t)$ is known.
3. If $\mu(y_t, \Theta^Q)$ and $\sigma(y_t, \Theta^Q)$ represent the drift and diffusion under a risk neutral measure Q, then the price of a contingent claim with payoff $G(y_{t+s})$ is available up to an integral equation.

Second, we propose a non-linear model to describe the VIX index and the variance risk premium. The VIX index is interpretable as an approximate market-implied estimate of the one-month ahead conditional volatility of the S&P 500 index conditional log-return under the *risk neutral* measure. So if $\ln R_{t:t+22}$ denotes the cumulative logarithmic twenty-two business day return on the S&P 500 index, the VIX index is a market implied estimate of $VIX_t^2 = \mathrm{Var}_t^Q(\ln R_{t:t+22})$. In our empirical application, we model the VIX (or squared VIX) as a non-linear diffusion through our transformation such that $y_t = VIX_t^2$. We report MLE and GMM estimates of the non-linear diffusion specification for the squared VIX.

We also study the variance risk premium. This risk premium, referred to as the variance (or volatility) risk premium, is defined as a difference between risk-neutral and objective measure volatility.

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Specifically, the standard definition (see for example Bollerslev et al., 2009)

$$VP_{t,\tau} = \mathrm{Var}_t^Q \{\ln R_{t:t+\tau}\} - \mathrm{Var}_t \{\ln R_{t:t+\tau}\}. \tag{2}$$

The variance risk premium is almost always positive, and is of a magnitude that has some important financial market implications, including "over-pricing" of S&P 500 put and call options.[1]

Our measure of the variance risk premium is not based directly on estimates of the two variances in (2), as is the common approach in the extant literature. Rather, our estimator is based on the difference in the drift for the squared VIX under the objective and risk-neutral measures. We show that our definition of the market-price-of-variance risk extracted from futures prices is equivalent to the definition in the extant literature which is based on Eq. (2).

There is a large literature on asset pricing using affine processes (see (Singleton, 2006) for an extensive review). There exist few models based on non-linear processes, although there are some notable exceptions: Ahn and Gao (1999) use a non-affine model with drift $\mu(r_t) = \kappa(\theta - r_t)r_t$ and diffusion $\sigma(r_t) = r_t^{3/2}$ to model the short rate while maintaining a tractable formula for bond prices. This process obtains through a transform $r_t = 1/x_t$ where x_t is an affine (CIR) diffusion. We generalize their idea of obtaining an analytically tractable process through a non-linear transform and our extension nests the reciprocal function of Ahn and Gao. Feunou and Meddahi (2007) derive a class of non-affine model by generalizing the dynamics of affine model in the frequency domain. Lewis (2000) derives (semi) analytic solutions to a stochastic volatility model with a linear drift and a $\sigma^{3/2}$ volatility-of-volatility term. Xiu (2014) develops a closed-form approximation for European options based on Hermite polynomial expansion where the underlying follows non-affine process. Our paper is further related to recent work by Chen and Joslin (2012) who use Fourier transforms of affine process to obtain semi-analytical pricing. Finally, Bu et al. (2011) use copulas-based transforms of the OU and CIR process to carry out maximum likelihood estimation. They do not consider asset pricing applications.

The pricing of VIX futures is studied in Zhang and Zhu (2006), Zhu and Zhang (2007), Lin (2007), Zhang et al. (2010), Zhang and Huang (2010), and Sepp (2008). For example, Zhang and Zhu (2006) study the VIX futures pricing assuming that the underlying stock price follows the stochastic volatility process of Heston (1993). Lin (2007) and Sepp (2008) incorporate jumps in the variance process. Zhang et al. (2010) use a two-factor affine model. Indeed, the literature on VIX futures typically uses affine models. We will venture beyond the affine class. This logic is the same as in the literature on interest rates. That is, it is possible that volatility can mean revert more quickly when volatility is very low or very high.

A major advantage of the affine class is that it is straightforward to price VIX futures from assumptions about the volatility dynamics for the underlying S&P 500 index, or even if the level of the S&P itself is endogenous. When volatility dynamics follow a non-linear diffusion it is more complicated to make this connection. In this case, even hypothetical futures on the squared VIX index are non-linear functions of state-variables.

[1] It is suggestive of large negative returns to buyers of S&P 500 put options as is observed empirically by Coval and Shumway (2001), Driessen and Maenhout (2006), and Bondarenko (2003) among others. Bollerslev et al. (2009) and Zhou (2010) find that the variance risk premium significantly predicts future equity returns. Goyal and Saretto (2009) and Driessen et al. (2009) study the cross-section of option returns. Goyal and Saretto (2009) report that cross-sectional returns increase in the difference between option implied, risk neutral and objective measure volatility.

If the VIX itself follows a scalar non-linear diffusion with drift function μ the futures prices are expected values

$$F(VIX_t, T) = E_t^Q(VIX_T) = x_t + \int_t^T E_t^Q[\mu(VIX_s)]\,ds \tag{3}$$

under mild regularity. If the drift function μ is linear, the futures price is a linear function in VIX_t. On the other hand, if μ is non-linear, the futures price cannot be found directly. A primary methodological contribution of our paper is that we describe a method for constructing non-linear drift functions under the risk neutral measure implicit in futures prices. While we apply our methodology to the VIX futures market, it applies equally well to other markets where futures contracts implicitly define expected values of processes that follow non-linear diffusions.

Our empirical results are indicative of strong non-linearity in the estimated drift of the VIX index under both probability measures. Under the objective measure the VIX index mean-reverts more quickly from extremely high and extremely low levels than what a linear model predicts. There is relatively little mean reversion when the index is near its historical average. A third order polynomial drift seems to fit the data well.

We use our estimates of the variance risk premium as well as estimated differences in forecasts under the two measures to construct predictive return regressions. The regressions show that in using our definition of the variance risk premium we get relatively high R^2's in predicting stock market returns. In particular, we show that our measure provides increased forecasting ability at longer forecasting horizons than what has been reported in the existing literature.

The remainder of the paper is organized as follows: in the next two sections we outline the methodological contribution in extracting non-linear drift and diffusion functions from observed futures prices. Section 4 reports the results from estimation on the VIX spot and futures data. Section 5 concludes.

2. Methodology

Let $(\Omega, \mathcal{F}, P)$ denote a probability space with an information filtration $(\mathcal{F}_t)$. The idea is to consider a functional (non-linear) transform of a diffusion process, x_t, with known properties. Specifically,

Assumption 1. We assume that x_t is a stationary Markov process with domain $\mathcal{D} \subset \mathbb{R}^n$,

$$dx_t = \mu(x_t, \vartheta)dt + \sigma(x_t, \vartheta)dB_t, \tag{4}$$

where $\mu : \mathcal{D} \times \mathbb{R}^k \to R^n$ and $\sigma : \mathcal{D} \times \mathbb{R}^k \to R^n$ are the drift and diffusion functions which we assume to be parameterized by a k dimensional parameter ϑ. Let $\mu_i(x_t)$ and $\sigma_{i,j}(x_t)$ denote the ith element of the drift function and (i, j)th element of diffusion function, respectively.

Assumption 2. The conditional characteristic function $\phi_\tau(u) = E_t \exp(\mathbf{i}u'x_{t+\tau})$ for $u \in \mathbb{C}^n$ can be computed explicitly.

Assumption 3. The transition density $p(x_{t+\tau} \mid x_t, \vartheta)$ can be computed explicitly.

By "computed explicitly" we mean that the functions can be computed either analytically or through numerical methods that are computationally fast. Assumption 2 is needed for asset pricing purposes while Assumption 3 facilitates likelihood based inference. A candidate for x is the general affine class of diffusions considered by Duffie et al. (2000). The affine class generally offers analytical tractability of the generating functions (up to ODE's) and special cases offer analytical transition densities.