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Game contingent claims in complete and incomplete markets

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Abstract

A game contingent claim is a contract which enables both the buyer and the seller to terminate it before maturity. For *complete* markets Kifer [Finance and Stochastics 4 (2000) 443–463] shows a connection to a (zero-sum) Dynkin game whose value is the unique no-arbitrage price of the claim. But, for incomplete markets one needs a more general approach. We interpret the contract as a generalized non-zero-sum stopping game. For the complete case this leads to the same results as in Kifer [Finance and Stochastics 4 (2000) 443–463]. For the general case we show the existence of an equilibrium point under the condition that both the seller and the buyer have an exponential utility function. For other utility functions such a point need not exist in the context of incomplete markets.

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1. Introduction

A game contingent claim (GCC) is a contract between a seller A and a buyer B which enables A to terminate it and B to exercise it at any time $t \in \{t_0, \dots, t_k\}$ up to a maturity date $T = t_k$ when the contract is terminated anyway.

More precisely, let $(\Omega, F, P, (\mathcal{F}_t)_{t\in[0,T]})$ be a filtered probability space satisfying the usual conditions of right-continuity and completeness, and let $(X_{t_i})_{i=0,\dots,k}$, $(Y_{t_i})_{i=0,\dots,k}$, $(W_{t_i})_{i=0,\dots,k}$ be sequences of real-valued random variables adapted to $(\mathcal{F}_{t_i})_{i=0,\dots,k}$ with

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$Y_{t_i} \leq W_{t_i} \leq X_{t_i}$ for $i = 0, \ldots, k-1$ and $Y_{t_k} = W_{t_k} = X_{t_k}$. If A terminates the contract at time t_i before B exercises then A should pay B the amount X_{t_i}. The other way around, A should pay B only Y_{t_i}. If A terminates and B exercises at the same time, then A pays B the amount W_{t_i}.

Definition 1.1. Let $\mathcal{S}_i, i = 0, \ldots, k$, be the sets of all stopping times, respectively $(\mathcal{F}_t)_{t\in[0,T]}$ with values in $\{t_i, \ldots, t_k\}$.

The above contract can be formulated as follows. If A selects a cancellation time $\sigma \in \mathcal{S}_0$ and B selects an exercise time $\tau \in \mathcal{S}_0$, then A pledges to pay B at time $\sigma \wedge \tau$ the amount

$$R(\sigma, \tau) = X_\sigma I(\sigma < \tau) + Y_\tau I(\tau < \sigma) + W_\tau I(\tau = \sigma).$$

The frictionless financial market consists of d risky assets whose discounted price processes are modeled by the $\mathbb{R}^d$-valued semimartingale S and one riskless asset with discounted price process equal to 1. We denote by Θ a suitable space of admissible trading strategies to be specified later.

Example 1.2 (Israeli call option). An American style call option with strike price K where also the seller can terminate the contract, but at the expense of a penalty $\delta_{t_i} \geq 0$, i.e. $Y_{t_i} = (S_{t_i}^{(1)} - K)^+$, $X_{t_i} = (S_{t_i}^{(1)} - K)^+ + \delta_{t_i}$, and $W_{t_i} = (S_{t_i}^{(1)} - K)^+ + \delta_{t_i}/2$.

Such a game version of an American option is safer for an investment company which issues it, and so it can be sold cheaper than the corresponding American option. As pointed out in Kifer (2000), essentially any contract in modern life presumes explicitly or implicitly a cancellation option by each side which then has to pay some penalty, and so it is natural to introduce a buyback option to contingent claims, as well. A prominent example are convertible callable bonds. The holder can convert them into a predetermined number of stocks of the issuing firm, and the issuer can recall them, paying some compensation to the holder. These contracts were approached in the economic literature for the first time by Brennan and Schwartz (1977), Ingersoll (1977a,b). Optimal conversion and call policies were derived. For a practical example of a convertible callable bonds see McConnell and Schwartz (1986).

In a complete market (i.e. Y, W, X are replicable by trading in S) one can solve our problem independently of agents' preferences: A just wants to minimize $E_Q(R(\sigma, \tau))$ whereas B wants to maximize the same expression (Q is the unique equivalent martingale measure). Thus, we have a zero-sum Dynkin stopping game. It is well-known that such a game has a *unique* value, cf. Ohtsubo (1986). Kifer (2000) shows by hedging-arguments that this value is also the *unique* no-arbitrage price of the GCC. In other words, the expectation of the (discounted) payoff under the unique equivalent martingale measure is the variable to be maximized, respectively, minimized, and this ensures consistency with the principle of no-arbitrage. Consequently, one has to solve a classical Dynkin game.

In incomplete markets this argument fails because there is more than one equivalent martingale measure. It is possible to superhedge the claim and get an interval of no-arbitrage prices, but then the feature of a stochastic game gets lost.