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A risk reserve model for hedging in incomplete markets

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ABSTRACT

This paper presents a new approach to the pricing and hedging problem for contingent claims in incomplete markets. We assume that traders wish to maximize the expected final payoff of the hedging portfolio and the claims, and we avoid the use of utility functions. Instead, we model how traders are punished when taking excessive risks in practice. To do so, we introduce an extra reserve bank account, which earns a smaller rate of return than a standard deposit bank account. The reserve account should always contain a minimal amount of money, which depends on the risk that the trader's portfolio is exposed to. We focus on a specific example which uses option price sensitivities (the 'Greeks') to specify the risk. The resulting optimization problem can then be solved in a rather explicit form, and we show how the solution naturally leads to bid–ask spreads, prices which depend on the trader's current position and implied volatility smiles.

1. Introduction

In this paper we present a new approach to the pricing and hedging problem in incomplete markets. In incomplete markets it is not possible to specify prices of all contingent claims in terms of the prices of other tradeables that have already been priced, and therefore objective pricing is not possible. One possible approach to this problem is to define an alternative optimization criterion for traders. One may for example minimize the expected losses from a position (Föllmer and Leukert, 2000), the variance of hedging errors (Schweizer, 1992, 1996) or the probability of a hedge with leads to losses (Föllmer and Leukert, 1999).

It is also possible to define prices which depend on the individual risk preferences of traders. Usually such preferences are modeled by utility functions, which can be used to define a ranking of stochastic alternatives. However, as mentioned for example in Carr et al. (2001), despite thorough research made in the field of utility maximization and the fact that it is well-developed from a theoretical point of view, the utility approach is difficult to use in practice. This is mainly due to the difficulty of determining the utility functions of traders and investors.

We therefore choose a different method, in which we simply assume that individuals aim to maximize their expected profits from trading. In many standard models, the maximization of the expected profit would lead to unbounded strategies: since the expected return on risky assets is larger than the risk-free rate, borrowing more money from a bank account and investing it in stocks S will result in a higher positive expected profit. Thus the optimal strategy is buying an unlimited number of stocks, which results in an unlimited expected profit. It also results in an unlimited risk though.

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In our model the risk is managed by means of an extra bank account with an interest smaller than the risk-free interest rate, which should be thought of as a capital reserve against risk. Capital reserve is the amount of money a financial institution has to put aside to cover an eventual loss. For many institutions, this amount is specified by a regulator, such as the Basel Committee on Banking Supervision, in its Basel Capital Accords. In these accords market risks, the risk we are interested in for our research, is supposed to be measured using VaR, Value-at-Risk (Jorion, 2007). Although this technique is widely applied, it has been criticized for its strong dependence on the estimation of small probabilities (Pilpel and Taleb) and weak performance in the event of market crashes (Hua and Wilmott). We will therefore allow more general functions which measure the risk in portfolio, and which determine the minimal amount of capital reserve that should be prescribed based on this risk.

Modelling the capital reserve as an extra bank account with an interest rate smaller than the risk-free can be seen as a slightly stylized reflection of what is happening in real life. In general, financial institutions or individual traders and investors have a certain profit target. At the same time, they have to remain solvent and thus the amount of capital they can put at risk is limited. This defines a target rate of return for the investments. When a trader takes on a risky position, he ties up some of the institution's capital: some money must be put aside in case the trade makes a loss. Since this money has to be readily available, it is kept in a bank account or in some very liquid instruments with yields that cannot possibly match the institution's target rate of return. This means it is not a profitable enough investment, unless there is some compensation for the 'lost' return associated with the solvency reserve. Hence, from the institution's point of view whenever risk is taken, they should charge a compensation for it in the prices. This can be seen as a generalization of the 'Cost of Capital' or 'Economic Capital' approach used in insurance companies and banks.

The extra bank account of our model which contains the risk reserve and has a smaller interest rate than the risk-free rate is not an extra traded asset in the market, because at any time it has to contain a prescribed minimal amount of money depending on the trader's portfolio risk. The mechanism of risk management is simple. When the risk associated with a certain portfolio is larger, more money must be kept in the reserve bank account, so more money is lost because of the lower interest rate on that account.

Despite the absence of utility functions we may then introduce an indifference price as the amount of money which makes the optimally behaving trader indifferent between having a certain option in his or her portfolio (while paying or receiving the quoted price for it) or not having it in his portfolio. This was first proposed by Hodges and Neuberger (1989). We introduce relative prices with respect to an existing portfolio, as was done earlier by Stoikov (2006): we define buying and selling indifference prices that depend on the current trader's portfolio.

Although the model setup based on these ideas can be quite general, we assume a finite discrete time set and a binomial model for the underlying asset price process and we only consider portfolios of European options in this paper. We also choose a special form of the risk function based on the portfolio Greeks, the sensitivities of the option portfolio value with respect to certain key parameters. This enables us to get analytical solutions for the indifference prices. These indifference prices show a difference between bid and ask prices, and produce volatility smiles, which may be fitted to the market by changing the parameters of the risk function.

The outline of the paper is as follows. In the next section we formulate our model in a general form, and also present a few specific examples. In Section 3 we show how we can solve the optimization program that is associated with the method, and we use the results in Section 4 to define indifference prices. In Section 5 we prove results concerning the absence of arbitrage in our model, and conclusions and suggestions for further research are formulated in the last section.

2. The risk reserve model

In this section we define the market model and the set of admissible trading strategies.

We consider a discrete time set $\mathcal{T}=\{t_i : i=1\dots N\}$ with $0=t_0<t_1<t_2<\cdots<t_{N-1}<t_N=T$ for a fixed $T>0$. Let $(\Omega,\mathcal{F},(\mathcal{F}_n)_{n=0\dots N},\mathbb{P})$ be a finite probability space with filtration $(\mathcal{F}_n)_{n=0\dots N}$, where the $\sigma-$algebra $\mathcal{F}_n$ models the events up to time t_n. We assume $\mathcal{F}_0=\{\emptyset,\Omega\}$, $\mathcal{F}_N=\mathcal{F}$, and that $\mathbb{P}(\{\omega\})>0$ for all $\omega\in\Omega$. We use the notation $\mathbf{E}_k$ for conditional expectations given $\mathcal{F}_k$, and $\mathbf{E}$ is the same as $\mathbf{E}_0$.

In this market we can invest in a risky asset S, a stock, and two riskless bank accounts (or bonds) B and Z. The second bank account Z contains the capital reserve, which works as a cushion for the risk and is required by regulators to contain a minimal amount of money which depends on the risk in a trader's position. It has a lower interest rate $\tilde{r}$ than the ordinary bank account B.

We denote the value of assets A at time point t_k as A_k for shortness and we use the notation $\Delta A_k=A_{k+1}-A_k$. Our model for the market prices is

$$B_{k+1}/B_k=e^{r\Delta t_k},$$

$$Z_{k+1}/Z_k=e^{\tilde{r}\Delta t_k},$$

$$S_{k+1}/S_k=R_k,\quad k=1,\dots,N, \tag{1}$$

where $r>\tilde{r}$, B_0=Z_0=1 and S_0 is given, and the $(R_k)_{k=0\dots N-1}$ are independent random variables which we will specify later on. Note that the existence of an extra bank account with a lower interest rate does not lead to immediate arbitrage,