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Pricing contingent claims in incomplete markets when the holder can choose among different payoffs

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Abstract

We suggest a valuation principle to price general claims giving the holder the right to choose (in a predefined way) among several random payoffs in an incomplete financial market. Examples are so-called "chooser options" and American options with finitely many possible exertion times but also some life insurance contracts. Our premium is defined by the minimal amount the writer must receive at time zero such that for all possible decision functions of the holder, the writer's utility is at least as big as the utility he would have if he did not offer this contingent claim. The valuation principle is consistent with no-arbitrage and can be interpreted as a generalization of Schweizer's indifference principle (Schweizer [Insurance: Mathematics and Economics 28 (2001) 31–47]). We show that in a complete financial market or, in general, if the writer has an exponential utility function, our premium is the supremum over all "utility-indifference premiums" related to all fixed random payoffs we get by fixing the decision function of the holder. For every other utility function our premium can be even larger than this supremum.

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1. Introduction

We are interested in options where the holder purchases the right to choose (in a predefined way) among several random payoffs offered by the seller. Such options could be a chooser option having the feature that, after a specified period of time, the holder can choose whether the option is a call or a put (cf. for example Hull, 2000), or an American option that can be exercised at any time up to the expiration date (and so the discounted payoff depends on the stopping time). Another example is an installment option, i.e. a European option in which the premium is paid in a series of installments and the holder has the right to terminate payments at any payment date, but then the option matures automatically (cf. Karsenty and Sikorav, 1996). In an insurance context, it could be a pension scheme

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where the policy-holder reaching a special age can swap his right to a pension for a single payment. Such choices offered by an insurance contract are called "embedded options". Many examples for such "options" are given by Held (1999).

In all these cases the insurer (writer of the option) has the problem that he does not know at time 0 which random payoff the insured (holder of the option) is going to choose.

In the spirit of Schweizer (2001), we consider a general model combining financial market risk and traditional actuarial risk. **Hereafter, we just call the insurer/writer "she" and the insured/holder "he"**.

Let $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t\in[0,T]})$ be a filtered probability space satisfying the usual conditions of right-continuity and completeness, and let the $\mathbb{R}^d$-valued semimartingale $S = (S_t)_{t\in[0,T]}$ model the *discounted* price processes of the d risky assets available for trade. Θ is a suitable space of admissible trading strategies to be specified later.

Definition 1.1. Let u be her utility function. It is a mapping from the set of random variables into $\mathbb{R}$ that is monotone in the sense that $X \le Y$ P-a.s. implies $u(X) \le u(Y)$.
The classical actuarial variance principle would correspond to $u(X) = E_P(X) - \lambda \operatorname{Var}(X)$, $\lambda > 0$, but it is known *not* to be monotone. The monotonicity is necessary for our valuation principle to be consistent with no-arbitrage.

For pricing random payoffs in incomplete markets (Schweizer, 2001) introduces—in the most general form—an indifference principle in the framework of financial markets. The idea is as follows: she can decide whether she insures a risk B, an $\mathcal{F}_T$-measurable random variable, for a premium h or not. The utility-indifference premium is defined as the premium which makes her indifferent with regard to this decision. She also takes into consideration that she can perhaps (partly) hedge the risk.

Definition 1.2. h is called a "utility-indifference premium" if it satisfies

$$\sup_{\vartheta\in\Theta} u\left(c + h - B + \int_0^T \vartheta_t \,\mathrm{d}S_t\right) = \sup_{\vartheta\in\Theta} u\left(c + \int_0^T \vartheta_t \,\mathrm{d}S_t\right), \tag{1.1}$$

where c is her initial capital.

For the variance and the standard derivation principle, closed-form valuations for many practically relevant products combining financial and actuarial risk, as for example unit-linked life insurance contracts or so-called financial stop-loss reinsurance contracts, are given by Møller (2000), using general results of Schweizer (2001). For the exponential utility function, more recently Becherer (2001) (see Theorem 2.4.1), derives a recursive computation formula for the premium considering a model which consists of a complete financial market and additional independent actuarial risk observed at discrete points of time.

The aim of this paper is to generalize this concept to situations where the random payment is not fixed at the beginning, but during the policy term the holder can choose in a contractually predefined way between several scenarios.

In the first instance, we consider a model with only one predefined decision time at which the holder can choose among a finite number of payoffs (Section 2). In Section 3, we deal with American style contingent claims where the holder can stop the contract before maturity T. More technical lemmas are left to the Appendix A.

2. Choice among a finite number of payoffs

Let $\mathcal{B} = \{B_1, \dots, B_k\}$ be a set of contingent claims, i.e. each B_i is an $\mathcal{F}_T$-measurable positive random variable. He can choose among these k different payoffs at the predefined stopping time τ (using the information $\mathcal{F}_\tau$). This means that there is a set of permissible decision rules

$$\mathcal{D} = \{\delta : \Omega \to \{1, \dots, k\}, \mathcal{F}_\tau\text{-measurable}\}.$$