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An efficient computational intelligence approach for solving fractional order Riccati equations using ANN and SQP

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ABSTRACT

A new computational intelligence technique is presented for solution of non-linear quadratic Riccati differential equations of fractional order based on artificial neural networks (ANNs) and sequential quadratic programming (SQP). The power of feed forward ANNs in an unsupervised manner is exploited for mathematical modeling of the equation; training of weights is carried out with an efficient constrained optimization technique based on the SQP algorithm. The proposed scheme is evaluated on two initial value problems of the Riccati fractional order equation with integer and non-integer derivatives. Comparison of results with the exact solution, and with reference numerical methods demonstrates the correctness of the proposed methodology. Performance of the proposed scheme is also validated using results of statistical analysis based on a sufficiently large number of independent runs.

1. Introduction

A new numerical technique is presented here for solution of Riccati differential equations of arbitrary order; the generic form with initial conditions is given as:

$$\begin{aligned} &\frac{d^{\nu}u(t)}{dt^{\nu}} + A(t)u(t) + B(t)u^2(t) = C(t), \quad 0 < t \leqslant T, \\ &\frac{d^{n}u(0)}{dt^{n}} = c_n, \quad n = 0, 1, 2, \ldots, N-1, \end{aligned} \tag{1}$$

where $\nu > 0$ is the order, $\nu \in \Re$, $N = \lceil \nu \rceil$, $u(t)$ is the solution, $A(t)$, $B(t)$, and $C(t)$ are known functions, and c_n are constants defining the initial conditions.

Variants of Eq. (1) with different orders play a significant role in many applications. The Solitary wave solution of a nonlinear partial differential equation [1] and the one dimensional static Schrödinger equation [2] are addressed with the Riccati differential equation with integer order derivatives. Many classical and modern dynamical systems have been modeled with the Riccati equation using fractional order derivatives [3–5]. Various applications in applied science and engineering

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including robust stabilization, network synthesis, diffusion problems, optimal filtering, controls, stochastic theory and financial mathematics involve the use of the Riccati differential equation [6–10].

This motivates recent work on solution of the equation (of arbitrary order) using both analytical and numerical methods. These include the Adomian decomposition method (ADM) [11], variational iteration method (VIM) [12], Fractional Adams–Moulton Method (FAMM) [13], the modified Homotopy perturbation method (MHPM) [14], generalized differential transform method (GDTM) [15], Chebyshev wavelets method (CWM) [16], Homotopy analysis method (HAM) [17], and so on. Beside these, solution of fractional Riccati differential equations (FRDE) has also been proposed recently using the Legendre wavelet method [18], Haar wavelets [19] and the hybrid Taylor series expansion with MHPM [20]. Stochastic methods using feed-forward artificial neural networks (ANNs) optimized with genetic algorithms (GA) and particle swarm optimization (PSO) have also been exploited to solve FRDEs [21,22]. There is however still room for investigation into computational intelligence methods which can improve results in terms of accuracy and reliability, with lower computational time and better convergence.

Artificial intelligence techniques based on neural network models have been extensively used in various applied science and engineering problems. Examples include modeling for automatic normalization of multitemporal satellite images [23], solving thin plate bending problems [24], modeling of thermotransport phenomenon in metal alloys [25], prediction of roll force and roll torque in hot strip rolling processes [26], and automatic diagnosis of Hashimoto's disease [27], etc. Neural network models optimized with global and local search methodologies have been broadly used for solving linear and nonlinear differential equations [28–30]. Some recent applications of these methods are nonlinear singular systems associated with Emden–Fowler type equations [31], the first Painlevé equation [32], nonlinear Schrodinger equations [33], nonlinear MHD Jeffery Hamel flows [34], nonlinear Bratu type equations [35,36] and Troesch's boundary value problems [37] etc. These techniques can be extended to solve linear and nonlinear differential equations involving fractional order derivative terms [38,39]. Special fractional order systems based on Riccati and Bagley–Torvik equations are illustrative examples of application of such techniques [40,41].

In this paper, a new solution of the nonlinear Riccati differential equation of fractional order is presented using unsupervised feedforward ANNs optimized with the sequential quadratic programming (SQP) algorithm to model the equation. Training of weights of the network is carried out with the SQP algorithm which is an efficient and reliable constrained optimization solver. Statistical analysis of the proposed scheme is carried out to prove its validity and effectiveness. Comparative studies of the results obtained are made with other state of the art numerical techniques based on Adams–Bashforth–Moulton Method (ABFMM) [42,43], MHPM [14] and the enhanced Homotopy perturbation method (EHPM) [44].

This paper is organized as follows. In Section 2 some basic definitions and important relations for the fractional differential equation are given. Section 3 describes the numerical method ABFMM for solving RFDEs. Section 4 explains mathematical modeling of the equation using ANNs along with formulation of a fitness function. Section 5 discusses the learning methodology for the weights of the neural network using the SQP technique. The last section presents simulation results, comparisons with other techniques, and statistical analyses for two case studies.

2. Basic definitions

This section defines important relations to be used later in the manuscript. Fractional integrals and derivatives are represented in the literature in a variety of ways. Riemann–Liouville, Caputo, Erdélyi–Kober, Hadamard, Grünwald–Letnikov and Riesz provide definitions for some functions [45,46]. These definitions have their own advantages for different mathematical problems. Definitions for fractional derivatives given by Riemann–Liouville and Caputo are used in this article.

Definition 2.1 (*Riemann–Liouville fractional integral and derivative*). A fractional integral of order $v > 0$ is given as [47]:

$$\begin{aligned} I^{v}u(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{v-1}u(\tau)d\tau, \\ I^0u(t) &= u(t), \end{aligned} \tag{2}$$

where I is the integral operator. Accordingly its fractional derivative of order $v > 0$ is written as:

$$\frac{d^v u(t)}{dt^v} = \left(\frac{d}{dt}\right)^n (I^{n-v}u)(t) \quad (n-1 < v \leqslant n), \tag{3}$$

where n is an integer.

Definition 2.2 (*Caputo fractional derivative*). A modified definition introduced by Caputo [47,48] for fractional derivative of order v is given as:

$$\frac{d^v u(t)}{dt^v} = I^{n-v}\frac{d^n u(t)}{dt^n} = \frac{1}{\Gamma(n-v)}\int_0^t (t-\tau)^{n-v-1}u^{(n)}(\tau)d\tau \quad (n-1 < v \leqslant n), \tag{4}$$

where I^v is given in (2).