Economic Multi-stage Output Feedback NMPC using the Unscented Kalman Filter

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Abstract: Nonlinear Model predictive control (NMPC) is a popular control strategy for highly nonlinear chemical processes. The ability to handle safety and environmental constraints along with the use of an economic objective makes NMPC highly appealing to industries. The performance of NMPC depends strongly on the accuracy of the model. In reality, there always are plant-model mismatch and state estimation errors. Hence the NMPC controller must be robust to uncertainties in the model as well as against estimation errors. Among the several approaches presented in the literature, the scenario-tree based multi-stage NMPC approach is a non-conservative and efficient formulation. In this approach, the evolutions of the plant for different realizations of the uncertainties are considered as different scenarios and the optimization problem is formulated as a multi-stage stochastic programming problem with recourse. In this work, we consider multi-stage output feedback NMPC using the Unscented Kalman Filter (UKF) where the nonlinearities are represented using deterministically chosen sigma points for state estimation. In the control problem, we explicitly consider the UKF estimation equations to predict the future evolution of the system. The proposed approach is illustrated by simulation results of fed-batch chemical reactor with an economic cost function.

Keywords: Model-based control, Output feedback Nonlinear model predictive control, Robust control, Unscented Kalman Filter, Economic control objective.

1. INTRODUCTION

Nonlinear Model predictive control (NMPC) is an advanced process control strategy for the control of nonlinear systems. The control problem is formulated as an optimization problem over a finite prediction horizon with an economic objective or a reference tracking objective. In addition to the possibility of optimizing an economic objective online, handling constraints with ease makes this approach highly attractive. The performance of the controller depends crucially on the accuracy in the prediction of the plant evolution. Plant-model mismatch and estimation errors cause the prediction to be less accurate and may lead to poor performance of the controller and in some cases it can even lead to instability. A practically relevant NMPC controller must be robust to the uncertainties and disturbances and satisfy the constraints at all times. The most prominent robust NMPC schemes in the literature are the min-max approach described in Scokaert and Mayne [1998], the tube-based approach in Mayne et al. [2011] and the multi-stage NMPC approach from Lucia et al. [2013]. Min-max approaches minimize the worst case cost of the predicted evolution enforcing the fulfillment of the constraints for all the cases of the uncertainty for one optimal input trajectory. The tube-based approach of nonlinear systems uses two controllers, a nominal controller and an ancillary controller. For the nominal controller more stringent constraints than the original constraints are imposed and the task of the ancillary controller is to make sure that

the path of the plant remains close to the nominal path so that the original constraints are satisfied. In the multi-stage NMPC approach (see Lucia et al. [2013] and Lucia et al. [2012]), the possible future evolutions of the plant for different realizations of the uncertainties are considered as different scenarios of the problem. The important feature of this approach is that it takes future information about the realization of the uncertainty into account at every stage by admitting different control moves for different future scenarios that branch from different points. This makes the approach less conservative compared to open-loop min-max NMPC schemes. If the scenario tree is an exact representation of the future uncertainties, multi-stage NMPC provides the optimal solution under the given feedback information structure by solving an open-loop optimization problem. When the state vector is not measured at each sampling interval but only noisy measurements of some outputs are available, additional uncertainty about the current state as well as inexact information about the future states must be taken into account. Thus the controller needs to be robust to the estimation errors as well. Output based NMPC schemes have been researched extensively in the literature using the robust MPC schemes and accounting for the estimation errors as described e.g. in Rawlings and Amrit [2009], Findeisen et al. [2003], Lee and Ricker [1994]. Multi-stage Output feedback NMPC was presented for the first time in Subramanian et al. [2014] using an EKF for state estimation. The scheme was shown to be robust against parametric uncertainties and bounded estimation error. In this work, we formulate a multi-stage output feedback NMPC using the Unscented Kalman Filter (UKF) where the nonlinearities are better represented by using deterministically chosen sigma points. In this controller, we consider the UKF

estimation equations for the prediction of the future evolution of the system. The innovations give the new information from the measurement at each sampling time and are used to update the predicted state using the system model. We model the samples of the innovations as new scenarios in the scenario tree in addition to the parametric uncertainties and we use the UKF estimation equations for the evolution of the future states along with the covariance information. The proposed approach is shown to be robust to plant-model mismatches and to estimation errors. The approach is illustrated by simulation results of a fed-batch chemical reactor with an economic cost function. In what follows, the UKF is discussed in Section 2 followed by standard multi-stage NMPC and multi-stage output feedback NMPC in Section 3. In Section 4, a case study of a highly nonlinear system is discussed followed by results which validate the method in Section 5. After discussing the results, we conclude this paper in Section 6.

2. THE UNSCENTED KALMAN FILTER

The most commonly used technique in the field of nonlinear estimation is the Extended Kalman filter (EKF) mainly because it is easy to implement. However because of the approximation of the nonlinearities present in the system using the Jacobian for the propagation of covariance information, higher order information gets lost and this leads to a less precise estimate of the states and the error covariance. The Unscented Kalman Filter offers an alternate and an efficient way to estimate states without the linearization of the model as presented in Julier and Uhlmann [1997]. In this method, $2\,n_x + 1$ deterministically chosen sigma points are sampled from the initial confidence interval and are propagated in time using the system model. Here n_x is the number of states in the system. From the propagated sigma points, the mean of the state and the new covariance information is obtained. A nonlinear system is assumed to be given by

$$x_k = f(x_{k-1}, u_{k-1}) + q_{k-1}, \qquad (1)$$
$$y_k = h(x_k) + r_k, \qquad (2)$$

where x_k is the state vector, u_k is the input vector, q_k is the process noise at a given time step k and $f : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_x}$ is the model of the system with n_x being the dimension of the state and n_u being the dimension of the input. In the equation (2), y_k represents the output vector with the dimension n_y, $h : \mathbb{R}^{n_x} \to \mathbb{R}^{n_y}$ is the measurement model and r_k represents the measurement noise. The covariances of q_k and r_k are represented as Q_k and R_k respectively. The covariance of the state x_k is given as P_k. With the current state estimate $\hat{x}_k$ and the covariance matrix P_k, the $2\,n_x + 1$ sigma points $\hat{x}_{\sigma_k}$ are calculated first as follows:

$$\lambda = \alpha^2 (n_x + \kappa) - n_x, \qquad (3a)$$
$$S_k = \sqrt{((n_x + \lambda)P_k)}, \qquad (3b)$$
$$\hat{x}^0_{\sigma_k} = \hat{x}_k, \qquad (3c)$$
$$\hat{x}^i_{\sigma_k} = \hat{x}_k + S^i_k, \forall i = 1, ..., n_x, \qquad (3d)$$
$$\hat{x}^i_{\sigma_k} = \hat{x}_k - S^i_k, \forall i = n_x + 1, ..., 2\,n_x, \qquad (3e)$$

where λ here is a scaling parameter obtained by the tuning parameters α and κ. S_k is the square root of the scaled covariance matrix P_k which can be obtained using Cholesky factorization. S^i_k represents the ith row of the square root matrix S_k. The sigma points are then assigned different weights for the calculation of the mean and covariance of the resulting state estimate

(mean) and probability distribution. The associated weights are given below.

$$w^0_m = \frac{\lambda}{(n_x + \lambda)}, \qquad (4a)$$
$$w^i_m = \frac{1}{2(n_x + \lambda)}, \forall i = 1, ..., 2\,n_x, \qquad (4b)$$
$$w^0_c = \frac{\lambda}{(n_x + \lambda)} + (1 - \alpha^2 + \beta), \qquad (4c)$$
$$w^i_c = \frac{1}{2(n_x + \lambda)}, \forall i = 1, ..., 2\,n_x, \qquad (4d)$$

where $w^i_m, \forall i = 0, ..., 2\,n_x$ are the weights associated with the sigma points $\hat{x}^i_{\sigma_k}, \forall i = 0, ..., 2\,n_x$ for the calculation of the mean and $w^i_c, \forall i = 0, ..., 2\,n_x$ are the weights associated with the corresponding sigma points for the calculation of the covariance matrix and β is another tuning parameter to approximate the probability density function. With these details, the algorithm for the Unscented Kalman Filter is given as follows (Wan and Merwe [2000]). In the Algorithm 1, the weighted sum

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Algorithm 1 The Unscented Kalman Filter (UKF)

Require: $\hat{x}_{k-1}, u_{k-1}, y_k, P_{k-1}, Q_k, R_k, \alpha, \beta, \kappa$
1: **Calculate** $x^i_{\sigma_{k-1}}, \forall i = 0, ..., 2\,n_x$ as in equation (3)
2: **Calculate** $w^i_m, \forall i = 0, ..., 2\,n_x$ as in equation (4)
3: **Calculate** $w^i_c, \forall i = 0, ..., 2\,n_x$ as in equation (4)
4: **for** i=0 to $2\,n_x$ **do**
5:     $x^{i-}_{\sigma_k} = f(x^i_{\sigma_{k-1}}, u_{k-1})$
6:     $y^{i-}_{\sigma_k} = h(x^i_{\sigma_k})$
7: **end for**
8: $\hat{x}^-_k = \sum_{i=0}^{2\,n_x} w^i_m x^{i-}_{\sigma_k}$
9: $y^-_k = \sum_{i=0}^{2\,n_x} w^i_m y^{i-}_{\sigma_k}$
10: $P^-_k = \sum_{i=0}^{2\,n_x} w^i_c [x^{i-}_{\sigma_k} - \hat{x}^-_k][x^{i-}_{\sigma_k} - \hat{x}^-_k]^T + Q_k$
11: $P_{yy} = \sum_{i=0}^{2\,n_x} w^i_c [y^{i-}_{\sigma_k} - y^-_k][y^{i-}_{\sigma_k} - y^-_k]^T + R_k$
12: $P_{xy} = \sum_{i=0}^{2\,n_x} w^i_c [x^{i-}_{\sigma_k} - \hat{x}^-_k][y^{i-}_{\sigma_k} - y^-_k]^T$
13: $K_k = P_{xy} \cdot P^{-1}_{yy}$
14: $\hat{x}_k = \hat{x}^-_k + K_k(y_k - y^-_k)$
15: $P_k = P^-_k - K_k P_{yy} K^T_k$

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of all the propagated (time updated) sigma points gives the a priori estimate $\hat{x}^-_k$ and the a priori covariance P^-_k is calculated as given in step 10 of the algorithm. Then the Kalman gain K_k is calculated from the cross covariance matrix P_{xy} and the measurement covariance matrix P_{yy}. The measurement information is added to the a priori estimate with the Kalman gain K_k via the innovations $\nu_k = y_k - y^-_k$. This gives the current state estimate $\hat{x}_k$ and with the update of the a priori state covariance matrix to the a posteriori covariance matrix P_k in step 15 of the Algorithm 1, the state estimation is complete.

3. MULTI-STAGE OUTPUT FEEDBACK NMPC

3.1 Multi-stage NMPC

We first shortly review the main concepts of the multi-stage NMPC approach presented in Lucia et al. [2013, 2014a] and then we discuss the main contribution of this paper: the integration of the Unscented Kalman Filter in the output based multi-stage NMPC setting. In multi-stage NMPC, the evolution of the uncertainty is represented by a tree of discrete scenarios that branches at each sampling instance until the end of the