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A full-space conformal mapping for the calculation of series impedance of overhead transmission lines and underground cables

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ABSTRACT

This paper introduces a 2-dimensional conformal transformation scheme for the parameter extraction of an arbitrary overhead transmission line or underground cable in an unbounded lossy space. The quasi-magnetic Helmholtz equation is solved using the finite element method (FEM). A modified bilinear transformation is employed to transform the unbounded domain to a bounded domain that is more efficiently handled in FEM simulations. Due to conformality of the transformation, the magnetic stored energy, which is used to calculate the per-unit-length inductance matrix, **L**, is independent from the mapping factor while the longitudinal current and its corresponding dissipated energy, which is used to calculate the per-unit-length resistance matrix, **R**, are functions of the mapping factor. Numerical results for an overhead transmission line and a buried multiconductor cable are compared with those calculated using Carson's approximation and Wedepohl's formula from near DC to the high frequency range.

1. Introduction

Solution of unbounded field problems in the case of lossy and nonhomogeneous media is of particular interest in the simulation of multiconductor overhead transmission lines and buried power cables. Attempts to solve the problem of a thin wire line over lossy homogenous ground have resulted in analytically-based formulations providing accurate answers at low frequencies [1–3] as well as high frequencies [4]. Solution of the buried cable problem in the case of lossy media, on the other hand, requires more effort and approximate formulas are almost always used. Low frequency methods, which include earth conduction effects, are widely used in commercial power system simulation tools for the calculation of per-unit-length resistance, **R**, and inductance, **L**, matrices of transmission lines [5]. They do not, however, enable easy modeling of arbitrary conductor geometries, such as sector-shaped cables, nonhomogenous earth configurations, or complicated cases when the conductor is at the surface of the earth.

The finite element method (FEM) is a powerful tool for the computation of arbitrary nonhomogeneous problems but, similar to other domain based numerical methods, it suffers from the inability to directly model infinite spaces. Generally, the infinite space is truncated far from the system considered (as far as time and hardware limitations permit) using various boundary conditions. In transmission line parameter extraction applications, the space truncation may produce unacceptable errors in the calculation of **R** and **L**. Several techniques have been introduced to alleviate the truncation problem. Following Bettes [6], these techniques may be categorized into five different groups: (1) truncation, (2) ballooning, (3) infinite or mapped elements, (4) transformation, and (5) coupling FEM with an analytical method or other numerical method. An extensive review of FEM open boundary techniques for static and quasi-stationary electromagnetic field problems can be found in [7].

The approach presented in this paper falls under the transformation group. Although spatial transformation is a well-known mathematical concept which has been frequently used in the solution of electromagnetic problems, it is still of interest to electromagnetic researchers [8–10]. A spatial transformation (mapping scheme) converts the physical unbounded space to a finite domain by a geometrical mapping method. The mapping scheme may be divided into two main categories; conformal and non-conformal methods. Conformal mapping was employed by Nath and Jamshidi for the calculation of scalar field problems in two-dimensional bounded domains [11]. Wong and Ciric used such transformations to calculate axi-symmetric open boundary problems [12]. Conformal mapping has also been used for solving waveguide problems [13]. The important feature of conformal transformation is that the governing equation in static electric and magnetic problems (Poisson's equation) remains unchanged [14]. In most cases, the post-processing formulations, which are usually used to calculate the distributed parameters in the case of a transmission line, also

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remain unchanged [15]. In the case of solving the Helmholtz equation, modifications, to incorporate a mapping factor, are required [16]. In some mapping schemes (not necessarily conformal), only part of the physical domain, usually the unbounded exterior area surrounding the region of interest, is mapped to a finite geometry, such as a circle in the case of the Kelvin transformation [7] or a disk [17,18].

Bilinear transformation, which is a special case of linear fractional transformation [19], is a conformal mapping scheme which has been employed for the solution of the Laplace equation by converting a semi-open half-space problem to an equivalent closed region [14]. This scheme maps the half-space, $y > 0$, to the interior of a unit circle and is applicable to planar transmission lines over a perfectly conducting infinite ground plane, such as microstrip.

In this paper, a generalization of bilinear transformation is presented for full-space unbounded problems. Unbounded two-dimensional space is divided to two half-spaces and each half-space is mapped to a unit circular disk with coinciding boundaries. This new mapping is then employed in the FEM simulation of the two-dimensional quasi-stationary approximation of the Helmholtz equation for the transmission line problem in a full-space unbounded lossy media. In Section 2, the introduced unbounded bilinear mapping scheme is described. The governing equation and post-processing formulation that is applied to the magnetic energy relation for obtaining the transmission line per-unit-length series impedance are derived in Sections 3 and 4, respectively. Numerical simulation results are presented in Section 5, with comparison with results obtained using Carson's approximation [1], Wedepohl's formula [5], and analytical formulas.

2. Mapping scheme

The bilinear mapping, $f(\zeta)$, that transforms the upper half-space, $y > 0$, in Cartesian coordinates, to a unit circle is given by

$$f(\zeta) = \frac{1 + j\zeta}{1 - j\zeta}, \tag{1}$$

where, $\zeta = x + jy$ represents a point on the complex plane. In order to map the lower half-space, $y < 0$, to a second unit circle using the same scheme, the lower space is first mapped to the upper space (y is replaced by $-y$), and the same mapping scheme is applied. This is achieved by replacing ζ with ζ^*, where * denotes complex conjugate. Mathematically, the two mapped circles are co-incident, but numerical solution of the problem requires the circles to be separated. The two circles are translated and rotated such that the $y > 0$ circle is adjacent to the $y < 0$ circle with the two circles touching each other at the origin. Infinity points are located at the top of the upper circle and the bottom of the lower circle. The interface between the two upper and lower spaces, $y = 0$, is the common boundary of the two regions and is mapped to the boundary of both circles. The proposed mapping is achieved by using the following mapping functions. For the upper half-space, we use

$$w_1 = u_1 + jv_1 = f_1(\zeta) = -j\,[f(\zeta) - 1]; \quad y \geq 0, \tag{2}$$

or,

$$u_1(x, y) = \frac{2x}{(1+y)^2 + x^2}, \qquad v_1(x, y) = \frac{2\left(y^2 + y + x^2\right)}{(1+y)^2 + x^2},$$

and, for the lower half-space,

$$w_2 = u_2 + jv_2 = f_2(\zeta) = j\,[f^*(\zeta^*) - 1]; \quad y \leq 0, \tag{3}$$

or,

$$u_2(x, y) = \frac{2x}{(1-y)^2 + x^2}, \qquad v_2(x, y) = \frac{2\left(-y^2 + y - x^2\right)}{(1-y)^2 + x^2}.$$

Here, w_1 and w_2 represent points in the upper and lower mapped half-spaces in which u_i and v_i ($i = 1, 2$) are the real and imaginary parts, respectively. The original space and the mapped space are shown in Fig. 1.

Using a conformal mapping has the advantage that the value of a scalar potential at each point in the original space is equal to that at its corresponding point in the mapped space [19] and the governing equations remain unchanged or only need to be modified using a simple mapping factor. The procedure for obtaining a mapping factor and necessary post processing integrals are summarized in the Appendix A.

3. Formulation of the problem

In this paper, the quasi-stationary approximation of the Helmholtz equation is considered. However, the method described here is also valid for the general form of the Helmholtz equation. The quasi-stationary approximation with longitudinal currents is employed to calculate the per-unit-length resistance, **R**, and inductance, **L**, matrices.

We begin with the differential form of the Maxwell's equations in the frequency domain ($e^{j\omega t}$ dependence)

$$\nabla \times \mathbf{E} = -j\omega\mathbf{B}, \tag{4a}$$

$$\nabla \times \mathbf{H} = \sigma\mathbf{E} + j\omega\mathbf{D}, \tag{4b}$$

$$\nabla \cdot \mathbf{D} = 0, \tag{4c}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{4d}$$

where σ is the conductivity of medium. Electric and magnetic fields can be written using the electric scalar potential, V, and the magnetic vector potential, **A**, as,

$$\mathbf{E} = -\nabla V - j\omega\mathbf{A}, \tag{5a}$$

$$\mathbf{B} = \nabla \times \mathbf{A}. \tag{5b}$$

Combining (4b) and (5b) with (5a) yields

$$\nabla \times \frac{1}{\mu}\nabla \times \mathbf{A} + (\sigma + j\omega\varepsilon)(j\omega\mathbf{A} + \nabla V) = 0. \tag{6}$$

Taking the divergence of (5a) and using Coulomb's gauge ($\nabla \cdot \mathbf{A} = 0$) results in

$$\nabla^2 V = 0, \tag{7}$$

which is simply the Laplace equation. Eqs. (6) and (7) are the governing equations of the problem.

Under the two-dimensional quasi-stationary assumption with a uniform structure along the z axis, the per-unit-length resistance and inductance are associated with the longitudinal component of the current, i.e., z component. As a result, the z component of (6) is considered in this paper only. As shown in [20], under the quasi-stationary assumption, the electric scalar potential V varies linearly with z. In other words, the z component of ∇V, $\partial V/\partial z$, is a constant. Here, we use ΔV to represent the z component of ∇V which simply represents the excitation voltage per unit length. Under these assumptions and neglecting the displacement current in conductors, to find the per-unit-length resistance and inductance, we solve

$$\frac{1}{\mu}\nabla_t^2 A_z + (\omega^2\varepsilon - j\omega\sigma)A_z = \underbrace{\sigma\Delta V}_{J_z^{ex}}, \tag{8}$$