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PID controller design for fractional-order systems with time delays

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ABSTRACT

Classical proper PID controllers are designed for linear time invariant plants whose transfer functions are rational functions of s^α, where $0 < \alpha < 1$, and s is the Laplace transform variable. Effect of input–output time delay on the range of allowable controller parameters is investigated. The allowable PID controller parameters are determined from a small gain type of argument used earlier for finite dimensional plants.

1. Introduction

Fractional order system models have been widely studied over the past two decades (see e.g., [1–7] and their references), where stability analysis and controller design problems are studied. Another line of research in this context is the design of fractional order controllers, including fractional order PID controllers, for fractional order as well as rational (finite dimensional) systems [8–14].

Fractional order systems appear in various engineering applications, see, e.g., [15–20]. It is interesting to see that they might appear in two ways. First, through theoretical modeling of physical phenomena and second from frequency domain experiments when traditional integer order models do not fit the data (for instance when Bode diagrams do not show slopes of integer multiples of 20dB/decade [21]). Many fields are concerned. In electricity, models of polarization emittance of metal electrodes [22] as well as capacitor models (based on purely empirical Curie's law of 1889) [23] are of fractional type. In material sciences, fractional order derivatives are used to model visco-elastic materials [24], non-laminated ferromagnetic components [25] or magnetic core coils [21]. Other physical phenomena such as heat conduction [26] or flexible structures [27] give rise to transfer functions with fractional powers of s (typically square root of s).

The topic of the present work is the design of *classical* proper PID controllers for fractional order systems.

Many different PID controller design techniques are available for rational (finite dimensional) systems with time delays; e.g. [28–31]. In this paper, we extend the approach of [28] to fractional order systems with time delays.

The class of plants considered and the feedback control problem studied are defined in Section 2. The proposed PID controller design method is described in Section 3. A numerical example is given in Section 4, and concluding remarks are made in Section 5.

2. Problem definition

Consider the standard single input–single output feedback system shown in Fig. 1, where C is the controller to be designed for the plant P.

We assume that the plant is linear and time invariant. Its dynamical behavior is represented by the transfer function

$$P(s) = e^{-hs} \frac{G(s^\alpha)}{s^\alpha - p} \tag{1}$$

where s is the Laplace transform variable, $h > 0$ is the total input–output time delay, $\alpha \in (0, 1)$ is the fractional order, $p \geq 0$ ($p^{1/\alpha}$ being the location of the unstable pole of the plant), and $G(w)$ is a rational stable transfer function in the variable $w = s^\alpha$ with $G(p) \neq 0$ and $G(0) \neq 0$. Such a plant was considered with $h = 0$ in [25] when modeling non-laminated electromagnetic suspensions.

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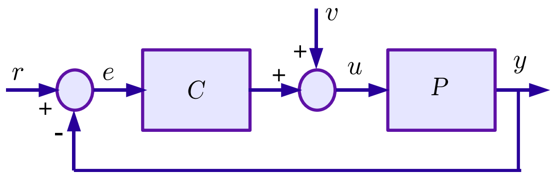

Fig. 1. Standard feedback system.

It is clear that we need $G(0) \neq 0$ for stabilizability of (1) by a controller which includes an integrator. We assume that α is a rational number, i.e., we are restricting ourselves to the class of *fractional systems of commensurate order*, [6]. There is a simple stability test for this type of systems, which can be seen below.

Given all the parameters of plant (1), our goal is to design a classical Proportional + Integral + Differential (PID) controller in the form

$$C(s) = K_p + \frac{K_i}{s} + K_d \frac{s}{\tau_d s + 1} \tag{2}$$

where K_p, K_i, K_d are free parameters and τ_d is an arbitrarily small positive number making the controller proper.

The feedback system formed by the controller C and the plant P is stable if $(1+PC)^{-1}$, $C(1+PC)^{-1}$ and $P(1+PC)^{-1}$ are stable transfer functions. These transfer functions are indeed fractional delay systems of retarded type and it has been proven [32] that H_∞-stability of these systems is equivalent to their BIBO-stability, a necessary and sufficient condition being that the system has no poles in the right half-plane (including no pole of fractional order at $s = 0$) and a numerical algorithm to test this property is available in [33]. In the case of fractional systems of commensurate order, checking stability can be done as follows (see e.g. [3,6]). Let $w = s^\alpha$ and assume that $T(w)$ is a rational function with poles $w_1, \ldots, w_n$. Enumerate the poles so that $w_1, \ldots, w_{2n_c}$ are complex conjugate, with $w_{n_c+k} = \bar{w}_k$ and $w_k = |w_k| e^{j\theta_k}$ where $\theta \in (0, \pi)$ for $k = 1, \ldots, n_c$, and $w_{2n_c+1}, \ldots, w_n$ are real. Then, the system $T(s^\alpha)$ is stable if and only if

$$\alpha \frac{\pi}{2} < \theta_k \quad \text{for } k = 1, \ldots, n_c, \text{ and}$$
$$w_k < 0 \quad \text{for } k = 2n_c + 1, \ldots, n.$$

We say that C is a stabilizing controller for the plant P if the feedback system formed by this pair is stable.

3. PID controller design

In this section, we design classical PID controllers in form (2) for plant (1). As in [28], the design will be done in two steps: first, PD controllers will be investigated, and then the integral action will be added.

3.1. PD controller design

A typical PD controller can be written in the form

$$C_{pd}(s) = K_p \left(1 + \widetilde{K}_d \frac{s}{\tau_d s + 1}\right). \tag{3}$$

We can express the non-delayed part of the plant as the ratio of two stable factors:

$$P(s) = e^{-hs} Y(s)^{-1} X(s) \quad \text{with } Y(s) := \frac{s^\alpha - p}{s^\alpha + x}$$
$$X(s) := \frac{G(s^\alpha)}{s^\alpha + x} \tag{4}$$

where $x > 0$ is the free parameter. While it is an arbitrary positive number at this stage, x plays an important role in the controller design.

With the notation introduced in (4), the feedback system stability is equivalent to stability of U^{-1}, where

$$U(s) := Y(s) + e^{-hs} X(s) C_{pd}(s). \tag{5}$$

Inserting C_{pd}, X and Y into (5), we have

$$U(s) = 1 - \frac{(p+x)}{s^\alpha + x} + e^{-hs} \frac{G(s^\alpha)}{s^\alpha + x} K_p \left(1 + \widetilde{K}_d \frac{s}{\tau_d s + 1}\right).$$

By choosing

$$K_p = (p + x) G(0)^{-1} \tag{6}$$

we obtain

$$\begin{aligned} U(s) &= 1 - \frac{(p+x)}{s^\alpha + x} \\ &\quad \times \left(1 - e^{-hs} G(s^\alpha) G(0)^{-1} \left(1 + \widetilde{K}_d \frac{s}{\tau_d s + 1}\right)\right) \\ &= 1 - \frac{(p+x)\, s^\alpha}{s^\alpha + x} \\ &\quad \times \left(\frac{1 - e^{-hs} G(s^\alpha) G(0)^{-1}}{s^\alpha} - \frac{\widetilde{K}_d e^{-hs} G(s^\alpha)\, s^{1-\alpha}}{G(0)\, (\tau_d s + 1)}\right). \end{aligned} \tag{7}$$

Since $\|\frac{s^\alpha}{s^\alpha + x}\|_\infty = 1$ for all $x > 0$, by the small gain theorem, U^{-1} is stable if

$$\left\| \left(\frac{1 - e^{-hs} G(s^\alpha) G(0)^{-1}}{s^\alpha}\right) - \widetilde{K}_d\, e^{-hs} G(s^\alpha) G(0)^{-1} \frac{s^{1-\alpha}}{\tau_d s + 1} \right\|_\infty < \frac{1}{(p+x)}.$$

The following results are immediate consequences of the above discussion.

Lemma 1. *For plant (1) there exists a stabilizing proportional controller, $C(s) = K_p$, if*

$$p < \left\| \frac{1 - e^{-hs} G(s^\alpha) G(0)^{-1}}{s^\alpha} \right\|_\infty^{-1} =: \psi_o. \tag{8}$$

When (8) holds, all proportional controllers in the form (6) are stabilizing, where x satisfies $0 < x < (\psi_o - p)$. □

Lemma 2. *Suppose there exist $\widetilde{K}_d \in \mathbb{R}$ and $\tau_d > 0$, such that*

$$p < \left\| \left(\frac{1 - e^{-hs} G(s^\alpha) G(0)^{-1}}{s^\alpha}\right) - \widetilde{K}_d\, e^{-hs} G(s^\alpha) G(0)^{-1} \times \frac{s^{1-\alpha}}{\tau_d s + 1} \right\|_\infty^{-1} =: \psi_d. \tag{9}$$

Then, the controller $C_{pd}(s) = K_p(1 + \widetilde{K}_d \frac{s}{\tau_d s + 1})$ is a stabilizing controller for plant (1) with $K_p = (p + x) G(0)^{-1}$ for all x satisfying $0 < x < (\psi_d - p)$. □

From the PD controller design method proposed in Lemma 2, we see that the allowable values of the proportional gain are in the range

$$K_p^{\min} := p G(0)^{-1} < K_p < \psi_d G(0)^{-1} =: K_p^{\max}.$$