Discrete Optimization 10 (2013) 61–68

Contents lists available at SciVerse ScienceDirect

Discrete Optimization

journal homepage: www.elsevier.com/locate/disopt

Weighted completion time minimization on a single-machine with a fixed non-availability interval: Differential approximability[☆]

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ARTICLE INFO

Article history:
Received 3 February 2012
Received in revised form 5 November 2012
Accepted 7 November 2012
Available online 8 December 2012

Keywords:
Differential approximation
Worst-case analysis
Scheduling
Weighted flow-time

ABSTRACT

This paper is the first successful attempt on differential approximability study for a scheduling problem. Such a study considers the weighted completion time minimization on a single machine with a fixed non-availability interval. The analysis shows that the Weighted Shortest Processing Time (WSPT) rule cannot yield a differential approximation for the problem under consideration in the general case. Nevertheless, a slight modification of this rule provides an approximation with a differential ratio of $\frac{3-\sqrt{5}}{2} \approx 0.38$.

1. Introduction

In this paper, we study the differential approximability of a well-known scheduling problem. Our work is motivated by the fact that the differential approximability concept has not yet been investigated for scheduling problems.

Contrary to the standard approximability based on the comparison in the worst case of a heuristic solution to the optimal one, the differential approximability principle consists in comparing the heuristic solution to both the optimal and the worst solutions. More precisely, we say that heuristic A is an α-differential approximation for problem Π if for every instance I of Π the following relation holds $f(A(I)) \leqslant \alpha f(opt(I)) + (1-\alpha)f(wst(I))$, where f is the objective function to be minimized in problem Π and the values $opt(I)$, $A(I)$ and $wst(I)$ denote respectively the values of an optimal solution, of an approximate solution and of a worst solution. The latter solution is defined as the optimal solution of a problem having the same instances and set of constraints as the initial problem but the opposite goal (i.e., max, if the initial problem is a minimization one and min if the initial problem is a maximization problem). Let us also note that worst solutions are not always easy to compute. For instance, for the minimization version of the travelling salesman problem, the worst solution is a Hamiltonian cycle of maximum total distance, i.e., the optimum solution of the maximum travelling salesman problem. The computation of such a solution is not trivial since the latter problem is as hard as the former one. On the contrary, examples of problems for which a worst solution is easily computed are maximum independent set where the worst solution is the empty set, minimum vertex cover, where this solution is the whole vertex-set of the input graph, or, even, minimum graph-colouring, where the worst solution consists in taking a colour per vertex of the input graph. The value α is called the differential ratio and it belongs to $(0, 1)$. For more details on these approaches, the reader is invited to consult Ausiello and Paschos [1], Demange and Paschos [2], and Hassin and Khuller [3].

☆ Research partially supported by the French Agency for Research under the DEFIS program TODO, ANR-09-EMER-010.

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doi:10.1016/j.disopt.2012.11.002

Let us note that the standard approximation ratio measures the relative position of the value $f(A(I))$ of a solution with respect to the optimal value $f(opt(I))$, while the differential approximation ratio measures the position of $f(A(I))$ in the interval $[f(opt(I)), f(wst(I))]$. In this sense, the two ratios provide different types of information about the problem under consideration and, in particular, produce different results for it. For instance, it is very well known that minimum graph-colouring is inapproximable within ratio $n^{1-\epsilon}$, for every $\epsilon > 0$ [4], while it is approximable within constant differential approximation ratio of 59/72 [5]. The same "asymmetry" occurs for the travelling salesman problem that is inapproximable within standard approximation ratio better than $2^{p(n)}$ for any polynomial p on the size n of the input graph (this is an easy extension of the corresponding inapproximability result given in [6]), while it is approximable within differential approximation ratio $3/4-\epsilon$, for any $\epsilon > 0$ [7]. On the opposite side, minimum vertex cover, although approximable within standard approximation ratio 2, it is inapproximable within differential ratio better than $n^{\epsilon-1}$, for every positive ϵ. From the above remarks, one can easily conclude that good or bad behaviour of a problem in the one or the other approximation framework, does say nothing about its behaviour in the complementary framework. Each of the frameworks induces its own results and conclusions about the approximability behaviour and properties of a given problem. Thus, let us say, an approximation scheme in one of the frameworks may be matched with a strong inapproximability result in the other one.

In the problem under consideration, we have a set of independent jobs to be performed on a single machine under the constraint of a fixed non-availability interval. The objective is to minimize the total weighted completion time under the non-resumable scenario. This problem has been proved to be NP-hard by Adiri et al. [8] and Lee [9] and it has been studied in the literature under various criteria. Several standard approximations have been proposed. A sample of them include the worst-case analysis of heuristic methods (see for example Adiri et al. [8]; Lee and Liman [10]; Sadfi et al. [11]; He et al. [12]; Wang et al. [13] and Breit [14]; Kacem and Chu [15]; Kacem [16]; Kellerer and Strusevich [17]). Efficient standard approximation schemes were also published in Kellerer and Strusevich [18]; Kacem and Mahjoub [19] and He et al. [12]. Other exact methods to solve this problem have been proposed in Kacem et al. [20,15,21]. For more details on scheduling problems under non-availability constraints, we refer the reader to the state-of-the-art papers by Lee [22] and Schmidt [23].

The review of the related literature shows that no differential approximation has been proposed to this problem according to the best of our knowledge. In a more general way, we did not find any work dedicated to the differential approximation to scheduling problems. For these reasons, this paper is the first successful attempt to develop a polynomial $\frac{3-\sqrt{5}}{2}$-differential approximation algorithm for the problem under consideration.

The paper is organized as follows. In Section 2, we present a description of the problem under consideration. Section 3 provides the differential analysis of the Weighted Shortest Processing Time heuristic (*WSPT*). In Section 4, we show that the modification of the above heuristic yields a differential ratio of $\frac{3-\sqrt{5}}{2}$. Finally, Section 5 concludes the paper.

2. Problem formulation

We have to schedule a set of n jobs $J = \{1, 2, \dots, n\}$ on a single machine. Every job i has a processing time p_i and a weight w_i. The machine is unavailable between t_1 and t_2 and it can process at most one job at a time. The length of the fixed non-availability interval length is denoted by Δt where $\Delta t = t_2 - t_1$. Let $C_i(\sigma)$ denote the completion time of job i in a feasible schedule σ. The aim is to find a schedule σ^* that minimizes the total weighted completion time $\sum_{i=1}^{n} w_i C_i(\sigma^*)$. With no loss of generality, we consider that all data are integers and that jobs are indexed according to the *WSPT* rule (i.e., $\frac{p_1}{w_1} \leqslant \frac{p_2}{w_2} \leqslant \dots \leqslant \frac{p_n}{w_n}$). Due to the dominance of the *WSPT* order (see Smith [24]), an optimal schedule is composed of two sequences of jobs scheduled in nondecreasing order of their indexes (one sequence will be performed before t_1 and another after t_2).

If all the jobs can be inserted before t_1, the problem under consideration $(\mathcal{P})$ has obviously a trivial optimal solution obtained by the *WSPT* rule (Smith [24]). We therefore consider only the problems in which all the jobs cannot be scheduled before t_1. Moreover, we consider that every job i can be inserted before the non-availability interval (i.e., $p_i \leqslant t_1$). Otherwise, it is obvious to schedule it after t_2.

In the remainder of the paper, we define $Z_k = \sum_{i=1}^{k} p_i$ for every $k \in \{1, 2, \dots, n\}$. Job g is identified by $Z_g \leqslant t_1$ and $Z_{g+1} > t_1$ (sometimes job $g+1$ is called the crossover job). Variable δ denotes the idle-time between t_1 and the completion time of g in the *WSPT* sequence (i.e., $\delta = t_1 - Z_g$). Moreover, $\varphi^*(\mathcal{P})$ denotes the minimum weighted sum of the completion times for problem $\mathcal{P}$ and $\varphi_\sigma(\mathcal{P})$ is the weighted sum of the completion times of schedule σ for problem $\mathcal{P}$.

It can be easily seen that the problem can be formulated by the following mixed integer quadratic model:

$$(\mathcal{P}) : \min \sum_{i=1}^{n} w_i C_i,$$

subject to:

$$C_i = x_i \left(\sum_{j=1}^{i} x_j p_j \right) + (1 - x_i) \left(t_2 + \sum_{j=1}^{i} \left(1 - x_j\right) p_j \right) \quad \forall 1 \leqslant i \leqslant n \tag{1}$$