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An approach to AHP decision in a dynamic context

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ABSTRACT

AHP (analytic hierarchy process) is used to construct coherent aggregate results from preference data provided by decision makers. Pairwise comparison, used by AHP, shares a common weakness with other input formats used to represent user preferences, namely, that the input mode is static. In other words, users must provide all the preference data at the same time, and the criteria must be completely defined from the start. To overcome this weakness, we propose a framework that allows users to provide partial and/or incomplete preference data at multiple times. Since this is a complicated issue, we specifically focus on a particular aspect as a first attempt within this framework. For that reason, we re-examine a mechanism to achieve consistency in AHP, i.e. a linearization process, which provides consistency when adding a new element to the decision process or when withdrawing an obsolete criterion under the dynamic input mode assumption. An algorithm is developed to determine the new priority vector from the users' new input. Finally, we apply the new process to a problem of interest in the water field, specifically, the adoption of a suitable leak control policy in urban water supply.

1. Introduction

AHP (analytic hierarchy process) is a multicriteria decision process [23] that involves aggregating various comparisons to obtain a priority vector that is representative of coherent results. In other words, AHP generates consolidated priorities about a number of alternatives that represent the will, likes, or decisions revealed by the preference data provided by one or more actors, or groups of actors, involved in the decision-making process. Achieving consistency in AHP has become an important issue [11,14,18,21,22] and different methods have been proposed [2,3,5–7,9,12,15,17,24,29].

AHP uses a specific input format for decision makers to express their preferences regarding multiple criteria and alternatives, namely, pair-wise comparisons. This format may be not perfect — yet it expresses user preferences reasonably well in many practical situations. After all the preference data has been collected, an algorithm is applied to generate consistent consolidated results.

However, a limitation of pair-wise comparisons is that before applying the decision model the experts must provide judgment data representing their preference with respect to all the elements involved. This kind of input is impossible in many practical situations. Consider, for example, the following two scenarios. Firstly, let us suppose that not all the elements for comparison are known or evident from the start. In leakage control, for example, only economic aspects have so far been widely considered. Nevertheless, environmental aspects have started to be considered as important, and even more recently, social elements have also started to play important roles in decision making on leakage control policies. A second scenario is when the consulted actors are unfamiliar with the effects of various items. As a result, it is difficult to collect complete preference information from decision makers at one time. It would be reasonable to allow decision makers to express their preferences at multiple times at their own convenience. In the meanwhile, partial results based on partial preference data could be generated from the data collected at multiple times — and this data could eventually be consolidated when the information is complete.

To consider the above mentioned scenarios, the input mode of the traditional AHP needs to be extended from a static mode to a dynamic mode. The dynamic mode involves the dimension of time. In other words, it will not be compulsory for users to provide input preference data at one point in time. A user will be able to input his/her preferences at multiples times. The user only needs to express his/her preference each time for a subset of elements, rather than the complete set. A change from static to dynamic mode will probably have many repercussions in future studies. It is impossible to address all of these issues at this time. Therefore, in this study we initiate a new approach and focus on a specific sub-problem. In this paper, we restrict ourselves to the case where a new criterion is added to the pool of previously considered criteria. This case can obviously be extended to the case of adding more than one criterion. The withdrawal of an obsolete criterion is readily obtained as a corollary.

The remainder of this paper is organized as follows. First, a short review of the linearization process [5] to achieve consistency is presented.

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In the methodology section, new results are presented that enable an efficient calculation of the new consistent matrix and its corresponding vector of priority after introducing a new criterion or withdrawing an obsolete criterion. Finally, the proposed methodology is applied to a real-world case in water leakage management, and conclusions are presented to close the paper.

2. Related work

In this section, we first review the pertinent literature on AHP and related work for the proposed methodology. We then provide a summary of our recent work and propose a method to achieve consistency for a non-consistent matrix based on a linearization procedure [5]. We have also extended this process to the case where a specific judgment should be changed [4]. In this paper we provide a new extension to consider the case in which a new decision element is introduced.

2.1. Some basics about consistency

Let us first recall the main facts about consistent matrices.

Let us consider an $n\times n$ real matrix A. A is positive if $a_{ij}>0$ for every i, j; A is homogeneous if $a_{ii}=1$ for every i; A is reciprocal if $a_{ij}=1/a_{ji}$, for every i,j. These are the typical properties of comparison matrices generally found in AHP. In addition, A is consistent if $a_{ik}=a_{ij}a_{jk}$, for every i, j, k. Among the different characterizations of consistent matrices, we recall the following which makes use of the mapping: J, associating to a positive matrix $A=(a_{ij})$ the matrix whose entry (i, j) is $1/a_{ij}$. If X is any matrix, then X^{T} denotes the transpose of X. Throughout this paper, it is assumed that the vectors of IR^n are column vectors.

The following result was established in a slightly different manner in [5].

Theorem 1. (Theorem 2.1, (ii) of [5]). A positive matrix A is consistent if and only if there is a vector $\mathbf{x}$ in IR^n such that $A=xJ(\mathbf{x})^{\mathrm{T}}$.

For a consistent matrix, the leading eigenvalue and the principal (Perron) eigenvector of a comparison matrix provide information to deal with complex decisions, the normalized Perron eigenvector giving the sought priority vector [22,23]. In the general case, however, A is not consistent. The hypothesis that the estimates of these values are small perturbations of the "right" values guarantees a small perturbation of the eigenvalues (see, e.g., [26]). For non-consistent matrices, the problem to solve is the eigenvalue problem $A\mathbf{w}=\lambda_{\max}\mathbf{w}$, where $\lambda_{\max}$ is the unique largest eigenvalue of A that gives the Perron eigenvector as an estimate of the priority vector. As a measurement of inconsistency, Saaty proposed using the consistency index $CI=(\lambda_{\max}-n)/(n-1)$ and the consistency ratio $CR=CI/RI$, where RI is the so-called average consistency index [23]. If $CR<0.1$, the estimate is accepted; otherwise, a new comparison matrix is solicited until $CR<0.1$.

2.2. Linearization process

From now on, $M_{n,m}$ will denote the set of $n\times m$ real matrices, $M^+_{n,m}$ the set of $n\times m$ positive matrices, and $tr(A)$ the trace of the matrix $A\in M_{n,n}$. It is well known that if we define $\langle A, B\rangle=tr(A^{\mathrm{T}}B)$ for $A, B\in M_{n,m}$, then $\langle\cdot,\cdot\rangle$ is an inner product. The derived norm from this inner product is customarily termed the Frobenius norm, and we denote it by $\|\cdot\|_{\mathrm{F}}$, i.e., $\|A\|_{\mathrm{F}}^2=tr(A^{\mathrm{T}}A)$. We define the map $L: M^+_{n,m}\rightarrow M_{n,m}$ associating it with a positive matrix $A=(a_{ij})$ whose (i, j) entry is $log(a_{ij})$. Its inverse mapping $E: M_{n,m}\rightarrow M^+_{n,m}$ associates a matrix $B=(b_{ij})$ with the matrix whose entry (i, j) is $exp(b_{ij})$.

The following map $\mathrm{d}: M^+_{n,m}\times M^+_{n,m}\rightarrow\mathrm{IR}$ defined by $\mathrm{d}(A,B)=\|L(A)-L(B)\|_{\mathrm{F}}$ is easily proven to be a distance. We propose using this distance in $M^+_{n,m}$ instead of the distance derived from the Frobenius norm. To motivate this proposal, note that we intend to solve approximation problems in $M^+_{n,m}$ and not in $M_{n,m}$; thus it is more natural to have a distance defined in $M^+_{n,m}$ than a distance defined in a larger set. Furthermore, let us consider the following example,

$$A_1=\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix},\quad B_1=\begin{bmatrix}1 & 2\\ 1/2 & 1\end{bmatrix},\quad A_2=\begin{bmatrix}1 & 8\\ 1/8 & 1\end{bmatrix},\quad B_2=\begin{bmatrix}1 & 9\\ 1/9 & 1\end{bmatrix}.$$

We have $\|A_1-B_1\|_{\mathrm{F}}\simeq 1.118$ and $\|A_2-B_2\|_{\mathrm{F}}\simeq 1.001$; which give the impression that the gap between A_1 and B_1 is similar to the gap between A_2 and B_2. This is not very intuitive because matrix A_1 reflects the fact that the two criteria are equivalent, while B_1 reflects that the second criterion is twice as important as the second criterion. Let us observe that the importance of the criteria in A_2 and B_2 are very close. Thus, in an intuitive point of view, the distance between A_1 and B_1 must be much greater than the distance between A_2 and B_2. Numerically, we have $\mathrm{d}(A_1, B_1)\simeq 0.9803$ and $\mathrm{d}(A_2, B_2)\simeq 0.1666$.

The linearization process derived in [5] states that the closest consistent matrix to an $n\times n$ comparison matrix A is given by the orthogonal projection of $L(A)$ onto

$$\mathcal{L}_n=\left\{L(A): A\in M^+_{n,n}, A\ \text{is consistent}\right\}. \tag{1}$$

Obviously, $X\in\mathcal{L}_n$ if and only if $E(X)$ is consistent. This subset $\mathcal{L}_n$ is a linear subspace of $M_{n,n}$ whose dimension is $n-1$. The orthogonal projection from $M_{n,n}$ to $\mathcal{L}_n$ will be denoted by $p_n: M_{n,n}\rightarrow\mathcal{L}_n$ and is given by a suitable Fourier expansion. In this expansion, use is made of the map given by

$$\phi_n(\mathbf{v})=\mathbf{v}1_n^{\mathrm{T}}-1_n\mathbf{v}^{\mathrm{T}},\qquad \mathbf{v}\in\mathrm{IR}^n, \tag{2}$$

where the symbol 1_n denotes the vector of IR^n having all its coordinates equal to 1: $1_n=[1\cdots 1]^{\mathrm{T}}\in\mathrm{IR}^n$ (this vector will play an important role in the sequel). We also use the standard inner product in IR^n (i.e., $\langle\mathbf{u},\mathbf{v}\rangle=\mathbf{u}^{\mathrm{T}}\mathbf{v}$) and the Euclidean norm in IR^n (i.e., $\|\mathbf{u}\|_2=(\mathbf{u}^{\mathrm{T}}\mathbf{u})^{1/2}$).

For the sake of clarity, we summarize the linearization theorem as follows:

Theorem 2. The subset $\mathcal{L}_n$ is a linear subspace of $M_{n,n}$ satisfying $\mathcal{L}_n=\mathrm{Im}\,\phi_n$ and $dim\mathcal{L}_n=n-1$. Furthermore, let $A\in M^+_{n,n}$.

(i) There exists a unique consistent matrix $Y\in M^+_{n,n}$ such that

$$\mathrm{d}(A,Y)\le\mathrm{d}\left(A,Y'\right)\ \forall Y'\ \text{consistent in}\ M^+_{n,n}.$$

This matrix Y is given by $Y=E(p_n(L(A)))$.

(ii) If $\{\mathbf{y}_1, \ldots, \mathbf{y}_{n-1}\}$ is an orthogonal basis of the orthogonal complement to $\mathrm{span}\{1_n\}$, then $\{\phi_n(\mathbf{y}_1), \ldots, \phi_n(\mathbf{y}_{n-1})\}$ is an orthogonal basis in $\mathcal{L}_n$,

$$\|\phi_n(\mathbf{y}_i)\|_{\mathrm{F}}^2=2n\|\mathbf{y}_i\|_2^2,\ \forall i=1,\ldots,n-1,$$

and the following matrix

$$\frac{1}{2n}\sum_{i=1}^{n-1}\frac{tr\left(L(A)^{\mathrm{T}}\phi_n(\mathbf{y}_i)\right)}{\|\mathbf{y}_i\|_2^2}\phi_n(\mathbf{y}_i)$$

is the orthogonal projection of $L(A)$ onto $\mathcal{L}_n$.

A simple computation shows that from Eq. (2), there is $[\phi_n(\mathbf{v})]^{\mathrm{T}}=-\phi_n(\mathbf{v})$ for any $\mathbf{v}\in\mathrm{IR}^n$, which, in view of Theorem 2, shows that any matrix in $\mathcal{L}_n$ is skew-Hermitian, in particular, $p_n(X)$ is skew-Hermitian for any $X\in M_{n,n}$.

The following result enables the discovery of an orthogonal basis of $\mathrm{span}\{1_n\}$ without any computations.