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A heuristic approach for deriving the priority vector in AHP

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ABSTRACT

This paper proposes an approach for deriving the priority vector from an inconsistent pair-wise comparison matrix through the nearest consistent matrix and experts judgments, which enables balancing the consistency and experts judgments. The developed algorithm for achieving a nearest consistent matrix is based on a logarithmic transformation of the pair-wise comparison matrix, and follows an iterative feedback process that identifies an acceptable level of consistency while complying with experts preferences. Three numerical examples are examined to illustrate applications and advantages of the developed approach.

1. Introduction

The Analytic Hierarchy Process (AHP) is one of widely used multi-criteria decision making (MCDM) methods [1]. It structures a decision problem as a hierarchical model consisting of criteria and alternatives. The priority vector needs to be derived from a pair-wise comparison matrix (PCM) that is collected from experts judgments. Extensive studies have been done on how to derive the priority vector from a PCM. For example, Saaty [1] proposed the eigenvector method (EM). However, the EM was always criticized from prioritization and consistency points of view. Therefore, some other methods have been developed to derive the priority vector from a PCM. Such as: Weighted least-squares method (WLSM) [2], Logarithmic least squares method (LLSM) [3], Least squares method (LSM) [4], Chi-square method (CSM) [5], Gradient eigenweight method (GEM) and Least distance method (LDM) [6], Geometric least squares method (GLSM) [7], Goal programming method (GPM) [8], Logarithmic goal programming approach (LGPA) [9], Fuzzy programming method (FPM) [10], Robust estimation method (REM) [11], Singular value decomposition approach (SVDA) [12], Interval priority method (IPM) called Possibilistic AHP for Crisp Data [13], Correlation coefficient maximization approach (CCMA) [14], Linear programming models (LPM) [15]. Moreover, Srdjevic [16] suggested combining different prioritization methods for deriving the priority vector; Wang [17] conducted an overview of methods for deriving the priority vector from a PCM. Besides, some comparative analysis of above mentioned methods for deriving the priority vector can be found in the literature [2,4,18–23]. It is concluded that there is no prioritization method that is superior to the others in all cases from the comparative analysis. Until now, the issue of their relative superiority is still unresolved though methods mentioned above are available for deriving the priority vector. This controversy was also addressed by Herman and Koczkodaj [24].

However, the priority vector derived from an inconsistent PCM depends strongly on the selected method. Therefore, different method may produce different priority vector. Based on this idea, we propose a new approach for deriving the

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priority vector that is based on the nearest consistent matrix and experts judgments. The proposed approach, firstly seeks the nearest consistent matrix by minimizing the distance between the given PCM and the required consistent matrix in the sense of the Frobenius norm metric, and then derives the priority vector through the nearest consistent matrix. This approach incorporates an extended version of the described linearization procedure [25–27], and is integrated with AHP for deriving the underling priority vector based on the revised PCM. Theorems and algorithms related with the proposed approach are developed in this paper. The proposed approach has the following advantages: (1) The algorithm for achieving the nearest matrix is fast and precise, and is easy to be implemented; (2) The tradeoff between the validity and the consistency is considered because the validity is very important in decision problems [28]; (3) The iterative feedback algorithm balances the consistency and experts judgments; (4) The iterative feedback algorithm has the convergence [29]; (5) The priority vector derived through the nearest consistent matrix is unique; (6) The proposed approach provides an alternative for deriving the priority vector from a PCM in AHP.

The rest of this paper is organized as follows. Section 2 presents a three-step algorithm for achieving a nearest consistent matrix from a given inconsistent PCM. Section 3 proposes a convergent iterative feedback algorithm to calculate the nearest consistent matrix. Section 4 concludes this paper.

2. Achieving the nearest consistent matrix

In general, it is difficult to obtain a consistent PCM, especially for the higher-order ones because of the limited ability of human thinking and the selected ratio scale. Therefore, it is a valid way to achieve the nearest consistent matrix which differs from the given inconsistent PCM as little as possible according to the assumed metric [25]. The related notations and mathematical concepts are described in the following.

2.1. Notations and mathematical concepts

The set of $n \times n$ matrices and the set of $n \times n$ matrices with positive entries are denoted by $M_{n,n}$ and $M^+_{n,n}$, respectively. The entry (i,j) of matrix A is denoted by $[A]_{ij}$ or a_{ij}. The matrix product component-wise (also called the Hadamard product) of two matrices A and B is denoted by $A \odot B$, where $[A \odot B]_{ij} = a_{ij}b_{ij}$.

The matrix norm is a mathematical index to measure the nearness of two given matrices [30]. For simplicity, the Frobenius norm is used and defined as

$$\|A\|_F = \left(\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^2\right)^{1/2}, \quad A \in M_{n,n}.$$

Furthermore, two inversed mappings are defined as

$$L : M^+_{n,n} \to M_{n,n}, \quad [L(X)]_{ij} = \ln([X]_{ij}),\ X \in M^+_{n,n};$$

$$E : M_{n,n} \to M^+_{n,n}, \quad [E(X)]_{ij} = \exp([X]_{ij}),\ X \in M_{n,n}.$$

The mapping L satisfies $L(X \odot Y) = L(X) + L(Y)$ for all $X,\ Y \in M^+_{n,n}$; The mapping E satisfies $E(X + Y) = E(X) \odot E(Y)$ for all $X, Y \in M_{n,n}$.

The distance between two given matrices A and B is denoted by the Frobenius norm and defined as

$$d(A,B) = \|A - B\|_F, \quad A,\ B \in M_{n,n}.$$

Definition 1. Matrix A is said to be positive reciprocal if $a_{ij} > 0$, $a_{ii} = 1$ and $a_{ij} = 1/a_{ji}$ for all $i,\ j \in \{1,2,\ldots,n\}$.

Definition 2 [28]. Matrix A is said to be Skew-Hermitian if $a_{ii} = 0$ and $a_{ij} = -a_{ji}$ for all $i,\ j \in \{1,2,\ldots,n\}$.

Definition 3. Matrix A is said to be consistent if $a_{ij} > 0$, $a_{ii} = 1$, $a_{ij} = 1/a_{ji}$ and $a_{ij} = a_{ik}a_{kj}$ for all $i,\ j,\ k \in \{1,2,\ldots,n\}$.

Definition 4 [25]. Matrix A is said to be L-consistent if $a_{ii} = 0$, $a_{ij} = -a_{ji}$ and $a_{ij} = a_{ik} + a_{kj}$ for all $i,\ j,\ k \in \{1,2,\ldots,n\}$.

Theorem 1. *Let $A \in M^+_{n,n}$, matrix A is positive reciprocal if, and only if, $L(A)$ is Skew-Hermitian.*

Proof. Since matrix A is positive reciprocal, by Definition 1, it follows that $a_{ii} = 1$ and $a_{ij} = 1/a_{ji} \iff [L(A)]_{ii} = \ln([A]_{ii}) = \ln a_{ii} = \ln 1 = 0$ and $[L(A)]_{ij} = \ln([A]_{ij}) = \ln a_{ij} = \ln(1/a_{ji}) = -\ln a_{ji} = -\ln([A]_{ji}) = -[L(A)]_{ji} \iff L(A)$ is Skew-Hermitian by Definition 2. $\square$