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Beckmann's transportation network equilibrium model: Its history and relationship to the Kuhn–Tucker conditions

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ABSTRACT

During 1952–54, Martin Beckmann, and his colleagues, formulated a nonlinear programming problem corresponding to behavioral assumptions from the viewpoint of an individual traveler concerning travel demand and cost-minimizing route choice over a congested road network. Their formulation was based on the conditions for a constrained maximum, recently derived by Kuhn and Tucker. This formulation was evidently the first time that economists used the Kuhn–Tucker conditions to formulate a new problem in economics, one of substantial practical importance and consequence, and quite possibly the first to use these conditions to formulate a new, large-scale problem in all fields of engineering. In this paper, an overview of the research leading to the formulation is offered. Then, the derivation presented in their monograph is described and explored in more detail. Finally, the impacts of this model on the field of transportation economics and the associated fields of transportation engineering and regional science are examined.

1. Introduction and objectives

While working at the Cowles Commission for Research in Economics at the University of Chicago during 1952–54, Martin Beckmann, C. Bartlett McGuire and Christopher B. Winston were engaged in a research project led by Tjalling C. Koopmans with the sponsorship of the Rand Corporation. The objectives of the project were to investigate problems of the 'allocation of resources' related to roads and railroads.[1] More specifically, McGuire stated: "Our original hope was that this work would give us some insight into the economics of city layout so that if a long-run policy of city dispersal were initiated, primarily for defense purposes, we could say something about where things should be dispersed to, and the costs or benefits thereof. While from this point of view I don't feel we have been very successful, I do think the work has led us to a better understanding of highway economics in general."[2]

In his Introduction to their monograph, Koopmans wrote: "… the present analysis can be useful as a starting point in developing a theory of balanced extension of the highway network, concurrently with industrial expansion or relocation. The increased vulnerability of metropolitan areas under modern warfare adds a note of urgency to the development of such a theory, already desirable before this complication arose."[3]

During this brief period, Beckmann, in collaboration with McGuire and Winsten, and led by Koopmans, formulated and analyzed the first model of origin-destination flows (demand) and user-equilibrium route flows for a congested road network (Beckmann et al., 1955, 1956). To accomplish this feat, they:

1. defined a representation of a road network with general link performance functions (travel cost as an increasing function of link flow) and conservation of flow constraints at nodes;
2. proposed a complementarity relationship for route flows from an origin to a destination, now known as the user-equilibrium conditions: if a route flow is positive, then its route cost must be a minimum; if a route cost is not a minimum, then its route flow must be zero;[4]
3. defined the general properties of origin-destination flow (demand) as a decreasing function of the endogenously determined route costs;
4. formulated a constrained maximization problem whose solution corresponded to the above demand and route choice behavioral assumptions; and

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[1] Beckmann related: "The ingenious title, 'Allocation of Resources,' gave Koopmans wide latitude. His main interest was in explaining Linear Programming's (Activity Analysis's) potential for revolutionizing the economics of location and transportation with initial focus on railroad systems and road networks" (Personal Communication, Dec. 4, 2007). See also, Hildreth (1986, p. 82).

[2] Letter from C.B. McGuire to Professor Oskar Morgenstern, Department of Economics, Princeton University, April 19, 1954.

[3] Beckmann et al. (1956, p. xvi).

[4] Slightly earlier, Wardrop (1952, p. 345) independently stated a similar criterion, which may be paraphrased as: all used routes have equal and minimal journey times.

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5. analyzed and interpreted the properties of this 'equilibrium' formulation and a related one concerning 'efficient' road networks.

Their formulation was evidently the first time that economists used the optimality conditions of Kuhn and Tucker (1951) as the basis for formulating an entirely new problem in economics, one of substantial practical importance and consequence.[5] They may also have been the first to use the Kuhn–Tucker conditions to formulate a large-scale model in any field of engineering, thereby making a seminal contribution to the emerging field of operations research as well as engineering generally.[6]

The record of how Beckmann and his colleagues achieved this breakthrough is relatively sparse: several earlier working papers offer no indication of the evolution of their thinking. Their book offers few insights into how the formulation came about. Moreover, they were in a hurry, as time and resources were limited.

When subsequently asked about the origins of the formulation of the model, Beckmann has generally stated that the model was joint work (of the three authors and Koopmans). When I interviewed him in 1996 and 1998, Beckmann held to this view. He also declined to discuss how they found the form of the objective function, a question discussed in Section 3.4, and often raised by scholars. However, when I put the same question to Bartlett McGuire in 1999, he immediately replied, "Of course, it was Martin's idea. We would sit together, and he would explain it to me."

In his Introduction, Koopmans stated: "Martin Beckmann, a mathematical economist especially interested in linear programming and economics activity analysis, contributed most of the chapters of the highway traffic analysis, with the exception of Chapter 1 on capacity." He went on to state, however: "This dry enumeration of contributions does not indicate the extent to which practically every chapter has been affected by the thinking of all members of the group." He also acknowledged that William Vickrey "has read the entire manuscript and given the authors the benefits of many detailed comments."

Earlier, I have authored or co-authored four papers concerning *A Study of Highway Transportation*, *Part I* of *Studies in the Economics of Transportation*: Boyce et al. (2005), Boyce and Nagurney (2006), Boyce (2007a, 2007b). The first of these reported on a panel discussion of the impact of the book at the 40th North American Meetings of the Regional Science Association International; the second reviewed the careers of McGuire and Winsten; the third provided historical details and references to the working papers of Beckmann and his co-authors, as well as sketching their model; and the fourth sought to offer a conjecture about the development of a new field using Beckmann et al. as an example. None of these papers explored their formulation in the depth attempted here, or offered conclusions on the significance of the work for science and engineering generally. Moreover, Anna Nagurney and I organized a special session of the 2005 Meeting of the Institute for Operations Research and the Management Sciences commemorating the 50th anniversary of the publication of the book, at which Martin Beckmann and Bartlett McGuire were honored with a citation from the Cowles Foundation.

My objective in this paper is to offer conjectures on how Beckmann, and his colleagues, formulated the network equilibrium model described in Chapter 3, Equilibrium, and Chapter 4, Efficiency. I seek to provide insights into how they derived this remarkable formulation, which may appear somewhat obvious today. Next, I review the Kuhn–Tucker conditions, and then show how they may have used them in formulating the model.

2. Preliminaries

The necessary and sufficient conditions for an inequality-constrained maximum are stated in Section 3, 'Lagrange multipliers for an inequality-constrained maximum,' of "Nonlinear Programming" by Kuhn and Tucker (1951). Using the notation of the authors, the problem is stated as follows:

Let $f_h(x)$, $h=1, \ldots, m$ be differentiable, nonnegative functions of $x=(x_i,\ i=1, \ldots, n)\geq 0$. Let $g(x)$ be a differentiable function of x for $x\geq 0$. Find x^0 that maximizes $g(x)$ constrained by $(f_h(x^0)\geq 0)$ and $x\geq 0$. For x^0 to be a local solution to the maximum problem, it is necessary that x^0 and some $u^0=(u_h,\ h=1,\ \ldots,\ m)\geq 0$ satisfy conditions (1) and (2) below, and a constraint qualification or regularity condition, which is satisfied if the constraints are linear.[7]

$$\left[\frac{\partial g(x^0)}{\partial x_i}+\sum_{h=1}^{m}u_h^0\frac{\partial f_h(x^0)}{\partial x_i}\right]\leq 0, i=1,\ldots n$$
$$x_i^0\left[\frac{\partial g(x^0)}{\partial x_i}+\sum_{h=1}^{m}u_h^0\frac{\partial f_h(x^0)}{\partial x_i}\right]=0, i=1,\ldots n$$
$$x^0\geq 0 \tag{1}$$

$$f_h(x^0)\geq 0, h=1,\ldots m$$
$$u_h^0[f_h(x^0)]=0, h=1,\ldots m$$
$$u^0\geq 0 \tag{2}$$

Subsequently, some scholars, including Kuhn (1976) himself, have noted that the Kuhn–Tucker conditions were proven earlier by Karush in a different context. Conditions (1) and (2) are now known as the Karush–Kuhn–Tucker conditions. In the following exposition, I will be primarily concerned with interpreting conditions (1) and (2), and not with the conditions for uniqueness of the results, which follow from establishing that the objective function being maximized is strictly concave.

Clearly, Beckmann himself understood the Kuhn–Tucker paper, perhaps in part from discussions with Morton Slater, and from Slater's paper, "Lagrange Multipliers Revisited" (Slater, 1950).[8] Beckmann also wrote a paper in 1952, "A Lagrangean Multiplier Rule in Linear Activity Analysis and Some Applications," which indicated that he was exploring the topic. At that time linear programming, or linear activity analysis as it was sometimes called, was the prevailing fashion. In his paper, Beckmann (1952) comments that non-linear activity analysis was 'controversial.'

A short exposition of the Kuhn–Tucker theorem is found in Section 4.3.2 of *Studies*, pp. 89–91. The fact that this account is in Chapter 4, rather than in Chapter 3 where the theorem was first used, suggests that Chapter 4 may have been written first; that is, they began with the analysis of efficient road networks, found that the optimality conditions implied an unexplained term, which they realized could be interpreted as an 'efficiency toll,'

[5] This statement is based on a careful search of economics journals, books and reports between 1950 and 1956. The first journal article found to refer to the Kuhn–Tucker theorem in formulating a new model was Chenery and Kretschmer (1956); the report by Beckmann et al. (1955) was issued by the Rand Corporation on May 12, 1955.

[6] This statement is based on bibliographies compiled by Rohde (1957), Wagner (1957), Riley and Gass (1958) and Dorn (1963), and a review of references contained therein.

[7] Several ways to state the Kuhn–Tucker conditions may be found in subsequent texts; here I use their original form.

[8] Slater, a mathematician, was a Research Associate at the Cowles Commission for Research in Economics during 1949–51. In this paper, he proposed a constraint qualification for the Kuhn–Tucker conditions, which was an issue at the time. Slater's unpublished paper was cited in more than 50 published papers, and in a comprehensive text on nonlinear programming by Bazaraa et al. (1993).