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An Abstract Extremal Principle with Applications to Welfare Economics

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In this paper we introduce general prenormal and normal structures in Banach spaces that cover conventional concepts of normals to arbitrary closed sets under minimal requirements. Based on these structures, we establish new abstract versions of the extremal principle in variational analysis, which plays a fundamental role in many applications. The main applications of this paper concern necessary conditions for Pareto optimality in nonconvex models of welfare economics. We obtain new results in this direction that extend approximate and exact versions of the generalized second welfare theorem for Pareto, weak Pareto, and strong Pareto optimal allocations.

Key Words: prenormal and normal structures; Banach spaces; variational analysis; normal cones and subdifferentials; extremal principle; Pareto optimality; welfare economics.

1. INTRODUCTION

It has been well recognized that the convex separation principle plays a crucial role in many aspects of nonlinear analysis, optimization, and their applications. In particular, a conventional approach to derive necessary optimality conditions in various optimization, optimal control, and equilibrium problems consists of applying convex separation theorems to either the convex sets in question or their tangential convex approximations.

This paper develops another approach to optimal solutions and related aspects of variational analysis that does not involve any convex approxima-

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tions and convex separation arguments. Instead, it is based on a different principle to study extremality of set systems using (generally nonconvex) normal cones in dual spaces that are not generated by primal tangential approximations. This approach, unified under the name of the *extremal principle* [27], goes back to the beginning of dual-spaced methods in nonsmooth variational analysis; see [26, 28] for more details, references, and discussions. Results obtained in this direction can be treated as variational extensions of the classical separation theorems to systems of nonconvex sets.

The primary goal of this paper is to obtain general versions of the extremal principle in terms of abstract prenormal and normal structures in Banach spaces. Then we apply these results to the study of Pareto optimal allocations in nonconvex models of welfare economics. Discussions of the results obtained and their comparison with the literature are presented in the subsequent sessions.

Our notation is basically standard. Let us mention that $B \subset X$ and $B^* \subset X^*$ stand, respectively, for the unit closed balls in the Banach space in question and its dual; $\overset{w^*}{\to}$ signifies the weak* convergence in X^*, and cl* denotes the weak* topological closure. Depending on the context, we use the notation Lim sup for either the *topological* Painlevé–Kuratowski upper (outer) limit

$$\operatorname*{Lim\,sup}_{x \to \bar{x}} F(x) := \operatorname{cl}^*\Big\{x^* \in X^* \big| \exists \text{ sequences } x_k \to \bar{x},\ x_k^* \overset{w^*}{\to} x^* \text{ with } x_k^* \in F(x_k),\ k \in \mathbb{N}\Big\} \quad (1.1)$$

of a set-valued mapping $F\colon X \rightrightarrows X^*$, or for its *sequential* counterpart when cl* is omitted in (1.1).

2. NORMAL STRUCTURES IN BANACH SPACES

In this section we consider abstract concepts of normals to arbitrary subsets of Banach spaces and designate minimal requirements to such concepts that allow us to derive fuzzy and exact versions of a general extremal principle, which is the main tool of our analysis and applications to welfare economics.

DEFINITION 2.1. Let X be a Banach space. We say that $\widehat{N}$ defines a *prenormal structure* in X if it associates, with every nonempty closed set