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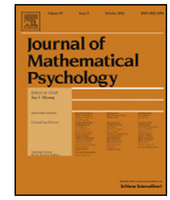

# Modeling geometric–optical illusions: A variational approach

Werner Ehm *, Jiří Wackermann

*Institute for Frontier Areas of Psychology and Mental Health, Freiburg i. Br., Germany*

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ABSTRACT

Visual distortions of perceived lengths, angles, or forms, are generally known as "geometric–optical illusions" (GOI). In the present paper we focus on a class of GOIs where the distortion of a straight line segment (the "target" stimulus) is induced by an array of non-intersecting curvilinear elements ("context" stimulus). Assuming local target–context interactions in a vector field representation of the context, we propose to model the perceptual distortion of the target as the solution to a minimization problem in the calculus of variations. We discuss properties of the solutions and reproduction of the respective form of the perceptual distortion for several types of contexts. Moreover, we draw a connection between the interactionist model of GOIs and Riemannian geometry: the context stimulus is understood as perturbing the geometry of the visual field from which the illusory distortion naturally arises. The approach is illustrated by data from a psychophysical experiment with nine subjects and six different contexts.


## 1. Introduction

"Geometric–optical illusions" (GOI) is a covering term for a broad class of phenomena, where visual perception of lengths, angles, areas or forms in a figure (e.g. a simple line drawing) is altered by other components of the figure. These phenomena demonstrate, generally, the dependence of a percept on its context, and allow to study the structural principles underlying the organization of visual percepts, or "laws of seeing" (Metzger, 1975). Since their discovery (Oppel, 1855, 1861), GOIs have been the subject of intensive experimental research (for comprehensive reviews see Coren & Girgus, 1978 and Robinson, 1998), but they are still far from being well understood. The variety of proposed explanations ranges from physiological theories, based on mutual interactions between elements of the neural substrate (e.g., retina or primary cortical areas: Carpenter & Blakemore, 1973; von Békésy, 1967; Walker, 1973), to purely mentalist theories, interpreting the GOIs as results of "unconscious inferences" (Helmholtz, 1867) or inappropriately applied cognitive strategies (Gregory, 1963). However, no unitary theory of the GOIs has been established until the present day, and it is even doubtful whether such a unified explanatory theory is conceivable (Coren & Girgus, 1978).

In the present paper we study a well-defined class of GOIs that are reducible to a common generating principle. The emphasis is not on the human vision system or on psychological factors, nor will physiological or psychological "mechanisms" be proposed; we aim at a representative–descriptive rather than explanatory–causal theory. Specifically, we focus on a class of GOIs in which perception of a *target* element – usually a segment of a straight line – appears distorted when presented with an array of (curvi)linear elements, in the following called the *context*. An example for such target–context interactions first reported by Hering (1861) is the illusory curvature of straight lines over which an array of concurrent lines is superposed (Fig. 1(a)). Since then a great number of GOIs have been constructed, discovered, or re-discovered on the same principle (Ehrenstein, 1925; Orbison, 1939; Wundt, 1898) (Fig. 1(b), (d), (e)).

These phenomena – hereafter called illusions of "Hering type" – are of particular interest for several reasons. First, they depend on *local* interactions between the target and the context elements, as is evidenced by variant figures in which parts of the context pattern are deleted (Wackermann, 2010). Next, they demonstrably do *not* depend on a "scenic" impression induced by the context patterns (Fig. 1(e), (f)). Finally, they all exhibit *angular expansion* at the target–context intersections: the illusory distortion of the target always acts to enlarge the acute angles at the intersection points (Fig. 1, passim). This effect, also dubbed "regression to right angles", seems to be constitutive for the class of GOIs of our interest (Berliner & Berliner, 1948; Horrell, 1971) as well as in other types of GOIs (Hotopf & Ollerearnshaw, 1972; Hotopf & Robertson, 1975).

These observations set up the framework for our modeling approach (Ehm, 2011). Starting with a minimal set of assumptions plus the fact that the straight line is the shortest path connecting two points, we propose a variational principle for the perception

* Correspondence to: Institute for Frontier Areas of Psychology and Mental Health, Wilhelmstr. 3a, D-79098 Freiburg i. Br., Germany.
*E-mail address:* ehm@igpp.de (W. Ehm).

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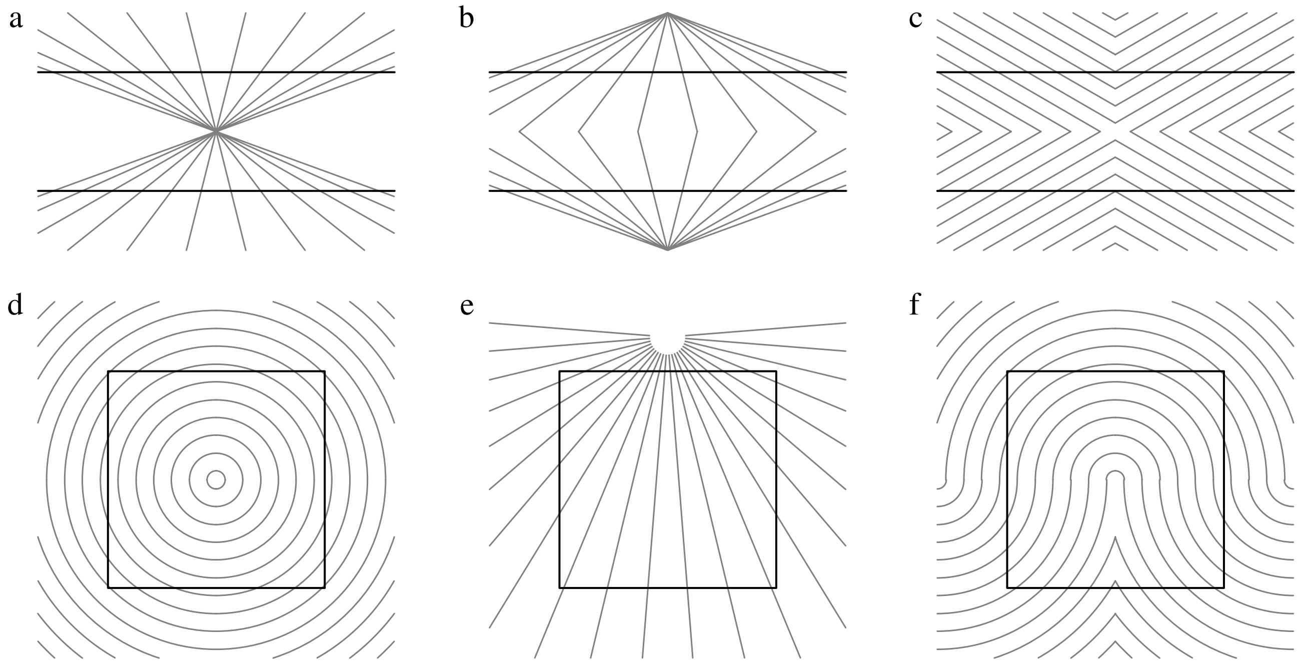


**Fig. 1.** Examples of geometric–optical illusions. Upper row: (a) Classic form of Hering's (1861) illusion and (b) its modification due to Thiéry and Wundt (1898); (c) Illusory bending of straight lines in a flat, non-perspectival context. Lower row: (d, e) Distortions of square shape in two different contexts (Ehrenstein, 1925; Orbison, 1939); (f) Trapezoid deformation of square shape similar to (e) in a different context, obtained by permutation of quadrants of pattern (d) (Wackermann, 2010).

of a linear target, draw a connection to Riemannian geometry, and show that approximate solutions of the respective variational problem reproduce the perceptual distortions of the target (Sections 2–4). Further, we report on a related psychophysical pilot experiment using six different context patterns (Section 5). Finally, we discuss achievements and limitations of this work (Section 6).

The main text covers the basic approach along with the applied methods and the results. All mathematical details, derivations, and proofs are given in the Appendix.

## 2. Variational problem for Hering type illusions

Our focus in this paper is on the case where the target is a straight line, and the context consists of a family of (generally curved) lines that intersect the target but not each other. We will conceive of the context lines as the stream lines of a planar flow given by a continuously differentiable vector field $v$ defined on some region $\Xi \subseteq \mathbb{R}^2$ containing the target in its interior.[1] To any point $\xi \in \Xi$ is attached a vector, $v(\xi)$, indicating the "velocity" of the flow at the point $\xi$. In view of the purely geometric character of the context it is natural to assume that $|v(\xi)| = 1$ for all $\xi \in \Xi$. Here $|a| = \sqrt{\langle a, a\rangle}$ and $\langle a, b\rangle = a_1b_1 + a_2b_2$ denote the Euclidean norm (length) and inner product, respectively, of vectors $a, b \in \mathbb{R}^2$. In geometrical terms, the normalized inner product $\langle a, b\rangle/|a|\,|b|$ gives the cosine of the angle between $a$ and $b$, which we denote as $\angle(a, b)$.

In graphical presentations of GOIs only a finite sample of context curves is displayed. The complete set of context lines, which form a continuum in the plane, may then be conceived of as continuously interpolating the sample. The target is here assumed to be the straight line, $\tau$, connecting two given endpoints $\tau_0, \tau_1 \in \mathbb{R}^2$.[2] In illusions of Hering type $\tau$ is not perceived as a straight line: it appears slightly curved. The basic idea of our approach is to model the deviating percept as a perturbation of $\tau$ that is characterized by a minimum principle. Setting up the principle involves three components:

(a) the local interactions hypothesis: the context $v$ "acts" only along candidate paths, in the vicinity of the target;
(b) the angular expansion hypothesis ("regression to right angles"), based on the phenomenology of GOIs (cf. Introduction);
(c) the fact that the straight line is the shortest path between two points.

Observing (b) and (c) we then posit the principle that, *given the context vector field* $v$, *the straight line target* $\tau$ *is distorted so that* (i) *the stream lines of* $v$ *(the context lines) are intersected "as orthogonally as possible", and* (ii) *the distorted line is as short as possible*.

This can be formulated mathematically as an optimization problem under side conditions. Since there is no *a priori* criterion suggesting length or orthogonality as the primary or the side condition, we propose to optimize a weighted mixture of the two terms. Specifically, we consider the following

**Variational problem** (VP). *Given the vector field* $v$, $t_0, t_1 \in \mathbb{R}$ *such that* $t_0 < t_1$, *endpoints* $\tau_0, \tau_1 \in \mathbb{R}^2$, *and some number* $\alpha \geq 0$, *minimize the functional*

$$J(x) = \int_{t_0}^{t_1} |\dot{x}(t)|\, dt + \alpha \int_{t_0}^{t_1} \frac{\langle \dot{x}(t), v(x(t))\rangle^2}{|\dot{x}(t)|}\, dt \qquad (1)$$

*over the set* $\mathcal{X}$ *of all twice continuously differentiable planar curves* $x \equiv \{x(t), t \in [t_0, t_1]\}$ *with given endpoints* $x(t_0) = \tau_0$ *and* $x(t_1) = \tau_1$ *such that* $x(t) \in \Xi$ *and* $|\dot{x}(t)| > 0$ *for every* $t$.

[1] For clarity it has to be emphasized that the vector field $v$ just serves us to represent the context; it is neither related to any kind of field theory or perceptive field, nor to the receptive field of the retina. Likewise, there is no supposition as to where the percept is located "materially". Finally, the term "flow" is used only metaphorically; it shall *not* convey any idea of motion.

[2] Subscripts may index objects (such as $\tau_0, \tau_1$) as well as the components of a vector (such that $y = (y_1, y_2)$ if $y \in \mathbb{R}^2$ is a row vector). The appropriate interpretation will always be evident from the context.