Performance Evaluation 69 (2012) 573–599

Contents lists available at SciVerse ScienceDirect

Performance Evaluation

journal homepage: www.elsevier.com/locate/peva

A *P*- and *T*-invariant characterization of product form and decomposition in stochastic Petri nets

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ARTICLE INFO

Article history:
Received 18 February 2011
Received in revised form 8 November 2011
Accepted 5 July 2012
Available online 13 July 2012

Keywords:
Stochastic Petri net
Product form
Decomposition
T-invariant
P-invariant

ABSTRACT

Structural product form and decomposition results for stochastic Petri nets are surveyed, unified and extended. The contribution is threefold. First, the literature on structural results for product form over the number of tokens at the places is surveyed and rephrased completely in terms of *T*-invariants. Second, based on the underlying concept of group-local-balance, the product form results for stochastic Petri nets are demarcated and an intuitive explanation is provided of these results based on *T*-invariants, only. Third, a decomposition result is provided that is completely formulated in terms of both *T*-invariants *and* *P*-invariants.

1. Introduction

Competition over resources is an important issue in many practical systems. Examples of such systems are computer systems, telecommunication networks, flexible manufacturing systems and hospitals, which typically consist of many departments and serve a wide variety of patient types. Pathways of patients are generally stochastic and various patient flows share different resources, of which operating rooms and diagnostic testing facilities are the most apparent. Typical questions arising are identification of bottlenecks, achievable throughput and maximization of resource utilization. Therefore, performance analysis is an important issue in the design and implementation of such real life systems.

Several approaches exist for performance analysis of complex systems, such as discrete-event simulation, numerical approximations or exact analytical results. Obtaining analytical results has two main advantages. First, it provides vital insight in the qualitative behaviour of involved systems, so that the key characteristics of a system can be detected. In particular, qualitative results related to the structure of the system are often of great importance. Second, it enables efficient computation of relevant performance measures. In many theoretical and practical studies of performance models involving stochastic effects, the statistical distribution of items (customers, jobs, etc.) over places (workstations, queues, etc.) is of great interest, since various performance measures can be computed from this distribution.

Three main formalisms exist for obtaining analytical closed form results for networks: queueing networks, stochastic process algebras and stochastic Petri nets. The selection of a specific formalism when studying a system preferably depends on the characteristics under investigation. Queueing networks are most suitable when the queueing structure at different locations in the network is the key aspect of the system. When a system consists of building blocks of different processes that are composed into a network, stochastic process algebras may be preferred. Stochastic Petri nets are appropriate when the flow of items and information through the network is the main feature of the system. When a specific formalism is applied, all network characteristics and all results are preferably formulated in the semantics of that formalism. In this paper we

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doi:10.1016/j.peva.2012.07.001

focus on Stochastic Petri nets, since we are interested in the interaction of flows within the system, such as naturally occur in hospital environments. All results are formulated in terms of the Petri net structure given by the P- and T-invariants, the central concepts in Petri Nets.

Composition and decomposition of closed form results contribute to less computational effort requirements and greater understanding of network behavior and performance. They allow for studying a system by analysing the characteristics of separate components. In this paper, we study closed form results for the equilibrium distribution of the number of tokens at the places of a stochastic Petri net and the decomposition of this equilibrium distribution into several components corresponding to subnets of the stochastic Petri net. Exact analytical results for the distribution of the number of items at places in performance models are in general very difficult to obtain. One of the most important analytical results for the equilibrium distribution describing the number of items at places in a performance model is the so-called *product form* equilibrium distribution found for a fairly wide class of theoretical queueing models. However, practical performance models seldom satisfy the product form conditions. Still, results obtained via the theoretical product form distributions are used for practical problems since these results are found to be robust, that is models which violate the product form conditions are often found to behave in a way very similar to a product form counterpart. The obvious advantages of these product form distributions are their simplicity, since the network behavior is captured in closed form in only a limited set of parameters. This makes product form solutions easy and powerful to use for computational reasons as well as for theoretical reflections for performance models involving congestion. Another important advantage of product form solutions is that it enables us to break down the analysis of a network in the analysis of separate components of the network.

It is widely believed that a form of *local balance* is the common element for all performance models with a product form equilibrium distribution. In this paper, *group-local-balance* is shown to be the concept identifying that the equilibrium distribution of a stochastic Petri net is of product-form nature. Boucherie and Van Dijk [1] presented the group-local-balance concept as the basis for the analysis of batch routing queueing networks. This paper provides a translation of these results into Petri net terminology. The results on the Markov chain level will then provide the foundation to discuss and further investigate structural Petri net implications. We survey the various structural results that are known for stochastic Petri nets with a product form equilibrium distribution over the number of tokens at the places [2–8]. The product form results for stochastic Petri nets known from the literature will shown to be unified by group-local-balance, as it forms the connecting principle between these results and the results known for batch routing queueing networks [1,9]. The results are derived and presented step-by-step to provide an intuitive understanding of the Petri net structure underlying the product form results.

The first structural product form results for stochastic Petri nets were presented by Henderson et al. [7]. These results are based on the assumption that a positive solution exists for a linear set of equations similar to the traffic equations for queueing networks. It will be shown that group-local-balance implies a positive solution to this linear set of equations, known as the *routing chain*, to exist. A characterization of the structure of the Petri net that is necessary and sufficient for the existence of a positive solution to the routing chain was provided by Boucherie and Sereno [2]. We show that this characterization implies that group-local-balance requires the stochastic Petri net to be an $\mathcal{SP}$-net [6], a stochastic Petri net in which each transition is covered by a minimal support T-invariant. Taking group-local-balance as a starting point enables us to provide additional structural implications and a more intuitive explanation of the known results. By formulating every result in terms of the Petri net structure given by the T-invariants, we also provide structural insights for results known at an algebraic level.

Finally, from the detailed understanding of the structure behind product form results, we are able to establish a decomposition result. This decomposition result is a generalization of the results obtained by Frosch and Natarajan [10,11] for closed synchronized systems of stochastic sequential processes, a class of Petri nets in which state machines are synchronized via buffer places. The decomposition result is completely formulated in terms of P- and T-invariants. Similar to buffer places, we define conflict places, which are places that are shared by different minimal closed support T-invariants. Using the P-invariants to assign conflict places as surplus places, places that can be omitted in characterizing the marking of the Petri net, we obtain an algorithmic procedure to verify whether product form holds and for decomposition of the stochastic Petri net into subnets. These subnets correspond to one or more common input bag classes, equivalence classes of T-invariants of the stochastic Petri nets that share an input bag.

Statement of contribution. Our contribution is threefold:

1. We survey the various structural results that are known for stochastic Petri nets with a product form equilibrium distribution over the number of tokens at the places and rephrases all these results in terms of T-invariants.
2. We unify and extend the product form results for stochastic Petri nets by showing that *group-local-balance* can be identified as the concept underlying all these structural results and we provide additional structural implications and an intuitive explanation of the known and new results, all based on T-invariants only.
3. We provide a decomposition result that is completely formulated in terms of both P- and T-invariants and their derivatives as defined in the paper: common input bag classes, conflict places and surplus places.

Outline. This paper is organized as follows. In Section 2, a detailed literature survey of product form results and decomposition is provided. For insight and self-containedness, a thorough introduction into the (stochastic) Petri net formalism is provided in Section 3. In Section 4, product form results for batch routing queueing networks based on the group-local-balance concept are translated into Petri net terminology. These results, presented on the Markov chain level,