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Faster algorithms for vertex partitioning problems parameterized by clique-width [☆]

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ABSTRACT

Many NP-hard problems, such as Dominating Set, are FPT parameterized by clique-width. For graphs of clique-width k given with a k-expression, Dominating Set can be solved in $4^k n^{\mathcal{O}(1)}$ time. However, no FPT algorithm is known for computing an optimal k-expression. For a graph of clique-width k, if we rely on known algorithms to compute a $(2^{3k}-1)$-expression via rank-width and then solving Dominating Set using the $(2^{3k}-1)$-expression, the above algorithm will only give a runtime of $4^{2^{3k}} n^{\mathcal{O}(1)}$. There have been results which overcome this exponential jump; the best known algorithm can solve Dominating Set in time $2^{\mathcal{O}(k^2)} n^{\mathcal{O}(1)}$ by avoiding constructing a k-expression Bui-Xuan et al. (2013) [7]. We improve this to $2^{\mathcal{O}(k \log k)} n^{\mathcal{O}(1)}$. Indeed, we show that for a graph of clique-width k, a large class of domination and partitioning problems (LC-VSP), including Dominating Set, can be solved in $2^{\mathcal{O}(k \log k)} n^{\mathcal{O}(1)}$.

Our main tool is a variant of rank-width using the rank of a 0–1 matrix over the rational field instead of the binary field.

1. Introduction

Parameterized complexity is a field of study dedicated to solving NP-hard problems efficiently on restricted inputs, and has grown to become a well known field over the last 20 years. Especially the subfields of Fixed Parameter Tractable (FPT) algorithms and kernelizations have attracted the interest of many researchers. Parameterized algorithms measure the runtime in two parameters; the input size n and a secondary measure k (called a parameter, either given as part of the input or being computable from the input). An algorithm is FPT if it has runtime $f(k)n^{\mathcal{O}(1)}$. Since we study NP-hard problems, we must expect that $f(k)$ is exponentially larger than n for some instances. However, a good parameter is one where $f(k)$ is polynomial in n for a large class of inputs. For a survey on parameterized complexity and FPT, we refer the reader to [13,12,22].

The *clique-width* of a graph G, introduced by Courcelle and Olariu [11], is the minimum k such that G can be expressed by a k-expression, where a k-expression is an algebraic expression using the following four operations:

- $i(v)$: construct a graph consisting of a single vertex with label $i \in \{1, 2, \dots, k\}$.

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- $G_1 \oplus G_2$: take the disjoint union of labeled graphs G_1 and G_2.
- $\eta_{i,j}$ for distinct $i, j \in \{1, 2, \dots, k\}$: add an edge between each vertex of label i and each vertex of label j.
- $\rho_{i \to j}$ for $i, j \in \{1, 2, \dots, k\}$: relabel each vertex of label i to j.

Clique-width is a well-studied parameter in parameterized complexity theory. It is therefore interesting to be able to expand our knowledge on the parameter and to improve on the preciseness of problem complexity when parameterizing by clique-width.

Courcelle, Makowsky, and Rotics [10] showed that, for an input graph of clique-width at most k, every problem expressible in $MSOL_1$ (monadic second-order logic of the first kind) can be solved in FPT time parameterized by k if a k-expression for the graph is given together with the input graph. Later, Oum and Seymour [25] gave an algorithm to find a $(2^{3k+2} - 1)$-expression of a graph having clique-width at most k in time $2^{3k}n^{\mathcal{O}(1)}$.[1] By combining these results, we deduce that for an input graph of clique-width at most k, every $MSOL_1$ problem is in FPT, even if a k-expression is not given as an input. However the dependency in k is huge and cannot be considered of practical interest. In order to increase the practicality of FPT algorithms, it is very important to control the runtime as a function of k.

If we rely on finding an approximate k-expression first and then doing dynamic programming on the obtained k-expression, we have two ways to make improvements; either we improve the algorithm that uses the k-expression, or we find a better approximation for clique-width. Given a k-expression, Independent Set and Dominating Set can be solved in time $2^k n^{\mathcal{O}(1)}$ [16] and $4^k n^{\mathcal{O}(1)}$ [3], respectively. Lokshtanov, Marx and Saurabh [19] show that unless the Strong ETH fails,[2] Dominating Set cannot be solved in $(3-\epsilon)^k n^{\mathcal{O}(1)}$ time even if a k-expression is given.[3] Hence, there is not much room for improvement in the existing algorithms when a k-expression is given.

There are no known FPT algorithms for computing optimal k-expressions, and the best known FPT algorithm for approximating an optimal k-expression via rank-width has an approximation ratio which is exponential in the optimal clique-width [23]. Therefore, even for the simple NP-hard problems such as Independent Set and Dominating Set, all known algorithms following this procedure has a runtime where the dependency is double exponential in the clique-width. The question of finding a better approximation algorithm for clique-width is an important and challenging open problem.

However, there is a way around this by avoiding a k-expression: Bui-Xuan, Telle and Vatshelle [5] showed that by doing dynamic programming directly on a rank decomposition, Dominating Set can be solved in $2^{k^2} n^{\mathcal{O}(1)}$ for graphs of clique-width k. Their algorithm with a runtime of $2^{\mathcal{O}(k^2)} n^{\mathcal{O}(1)}$ is not only for Independent Set and Dominating Set but also for a wide range of problems, called the *locally checkable vertex subset and partitioning problems* (LC-VSP problems). Tables 1 and 2 list some well known problems in LC-VSP.

In this paper we improve on these results by using a slightly modified definition of rank-width, called $\mathbb{Q}$-*rank-width*, based on the rank function over the rational field instead of the binary field. The idea of using fields other than the binary field for rank-width was investigated earlier in [18], but our work is the first to use $\mathbb{Q}$-rank-width to speed up an algorithm.

We will show the following:

- For any graph, its $\mathbb{Q}$-rank-width is no more than its clique-width.
- There is an algorithm to find a decomposition confirming that $\mathbb{Q}$-rank-width is at most $3k + 1$ for graphs of $\mathbb{Q}$-rank-width at most k in time $2^{3k} n^{\mathcal{O}(1)}$.
- If a graph has $\mathbb{Q}$-rank-width at most k, then every fixed LC-VSP problem can be solved in $2^{\mathcal{O}(k \log k)} n^{\mathcal{O}(1)}$-time.

This allows us to construct an algorithm that runs in time $2^{\mathcal{O}(k \log k)} n^{\mathcal{O}(1)}$ for graphs of clique-width at most k and solve every fixed LC-VSP problem, improving the previous runtime $2^{\mathcal{O}(k^2)} n^{\mathcal{O}(1)}$ of the algorithm by Bui-Xuan et al. [7].

We also relate the parameter $\mathbb{Q}$-rank-width to other existing parameters. There are several factors affecting the quality of a parameter, such as: Can we compute or approximate the parameter? Which problems can we solve in FPT time? Can we reduce the exponential dependency in the parameter for specific problems? And, how large and natural is the class of graphs having a bounded parameter value?

This paper is organized as follows: In Section 2 we introduce the main parts of the framework used by Bui-Xuan et al. [7], including the general algorithm they give for LC-VSP problems. Section 3 revolves around $\mathbb{Q}$-rank-width and is where the results of this paper reside. We show how $\mathbb{Q}$-rank-width relates to clique-width, and reveal why we have a good FPT algorithm for approximating a decomposition. In Section 4, we give our main result, which is an improved upper bound on solving LC-VSP problems parameterized by clique-width when we are not given a decomposition. We end the paper with Section 5 containing some concluding remarks and open problems.

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[1] Later, Oum [23] obtained an improved algorithm to find a $(2^{3k} - 1)$-expression of a graph having clique-width at most k in time $2^{3k} n^{\mathcal{O}(1)}$.

[2] The Strong Exponential Time Hypothesis (Strong ETH) states that SAT cannot be solved in $\mathcal{O}((2-\epsilon)^n)$ time for any constant $\epsilon > 0$. Here n denotes the number of variables.

[3] Their proof uses pathwidth, but the statement holds since clique-width is at most 1 higher than pathwidth.