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ScienceDirect
Games and Economic Behavior 61 (2007) 198–224
GAMES and Economic Behavior
www.elsevier.com/locate/geb

The envelope theorem for locally differentiable Nash equilibria of finite horizon differential games

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Received 4 May 2006
Available online 23 March 2007

Abstract

Envelope theorems are established for a ubiquitous class of finite horizon differential games. The theorems cover open-loop and feedback information patterns in which the corresponding Nash equilibria are locally differentiable with respect to the parameters of the game. Their relationship with extant envelope results is discussed and an application of them to a generalized capital accumulation game is provided. An important implication of the theorems is that, in general, the archetypal economic interpretation of the costate vector, namely, as the shadow value of the state vector along the Nash equilibrium, is valid for feedback Nash equilibria, but not for open-loop Nash equilibria.

JEL classification: C60; C61; C73

Keywords: Envelope theorem; Differential games; Nash equilibria

1. Introduction

The envelope theorem, first introduced into economic theory by Hotelling (1932, p. 594) in the context of a model of a profit maximizing firm facing a competitive market for its inputs, is now so fundamental to the modern development of microeconomic theory that its importance is difficult to overstate. It was Samuelson (1947, p. 34), however, who provided the first proof of the envelope theorem for the generic class of differentiable, unconstrained optimization problems. But it took twenty-four more years before a proof of the envelope theorem for the class of

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0899-8256/$ – see front matter
doi:10.1016/j.geb.2007.01.004

differentiable, constrained optimization problems was provided by Afriat (1971, pp. 355–357). It seems likely, nonetheless, that several of the prominent mathematical economists of the 1950s and 1960s had contemplated the extension of the envelope theorem to differentiable, constrained optimization problems. About the same time, Hadley and Kemp (1971, pp. 117–119) derived envelope results with respect to the limits of integration and the fixed state endpoints for a general calculus of variations problem. Shortly thereafter, Oniki (1973, p. 278) asserted the existence of, but did not prove, an envelope result for a general parameter in optimal control problems. Rather, it was Epstein (1978, pp. 119–120) who first derived an envelope expression for a general parameter in optimal control problems. Seierstad (1982) provided the generalization for the limits of integration and the fixed state endpoints for a general control problem.

What is surprising, especially given the intense research that took place in the 1970s and 1980s in game theory, is that 14 more years passed before the envelope theorem was further extended by Caputo (1996) to cover static games with locally differentiable Nash equilibria. In view of the analytical difficulties of obtaining closed-form solutions to such games in all but the simplest cases, one might have conjectured that the qualitatively powerful and empirically rich envelope theorem would have been extended to cover this paradigm more rapidly.

With these observations in mind, the goal of this paper is to extend the envelope theorem even further, so as to be applicable to a generic class of differential games with various information patterns which posses locally differentiable Nash equilibria. Inasmuch as this is a natural extension of the aforementioned literature, little motivation need be provided. Nevertheless, two remarks seem worthwhile at this juncture. First, given that differential games are even more difficult to solve for a closed-form solution than optimal control problems or static games, the results obtained herein go some way towards overcoming the fact that "It is difficult to obtain results from general differential games" (Reinganum, 1982, p. 674), seeing as the envelope results are secured in a general setting. Second, the theorems uncover an unanticipated result, to wit, that the "well-known" economic interpretation of the costate vector is, in general, invalid for one of the two information patterns contemplated. In particular, only in the case of a feedback information structure and its associated feedback Nash equilibria does the costate vector have the archetype economic interpretation of the shadow value of the state vector along the Nash equilibrium path, just as it does in optimal control theory.

Two different information structures and their corresponding Nash equilibria are considered. Following the taxonomy of Fershtman (1987) and Basar and Olsder (1999), envelope theorems are developed for

(i) a *feedback information structure* and its corresponding feedback Nash equilibria, and
(ii) an *open-loop information structure* and its associated open-loop Nash equilibria.

An application of the theorems in the context of a generalized capital accumulation game rounds out the contribution of the manuscript, and at the same time, imparts some economic intuition to the results. The corresponding results for a *memoryless perfect state information structure* and its closed-loop no-memory Nash equilibria may be found in Caputo (2006).

2. Technical preliminaries

To begin, consider the ubiquitous class of finite horizon differential games consisting of a finite number $P \in \mathbb{Z}_{++}$ of players, indexed by $p = 1, 2, \ldots, P$, and played over a finite time horizon $[t_0, t_1] \subseteq \mathbb{R}$. The state of the differential game at each instant $t \in [t_0, t_1]$ is given by the