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# Computing Nash equilibria by iterated polymatrix approximation

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**Abstract**

This article develops a new algorithm for computing Nash equilibria of $N$-player games. The algorithm approximates a game by a sequence of polymatrix games in which the players interact bilaterally. We provide sufficient conditions for local convergence to an equilibrium and report computational experience. The algorithm convergences globally and rapidly on test problems, although in theory it is not failsafe because it can stall on a set of codimension 1. But it can stall only at an approximate equilibrium with index +1, thus allowing a switch to the global Newton method, which is slower but can fail only on a set of codimension 2. Thus, the algorithm can be used to obtain a fast start for the more reliable global Newton method.


*JEL classification:* C63; C72

*Keywords:* Game; Equilibrium; Algorithm; Homotopy; Polymatrix approximation

## 1. Introduction

The standard algorithms for calculating Nash equilibria of $N$-player games use homotopy methods to trace equilibria in the graph of the Nash correspondence. The equilibria are traced above a path of games from a starting point that is a game whose equilibrium is known, to the terminal point that is the target game whose equilibria are wanted (Eaves, 1972; Eaves and Schmedders, 1999). The most effective implementation recently is the algorithm of Herings and Peeters (2001, 2002) that uses a differentiable homotopy to follow the path of Harsanyi's (Harsanyi, 1975) tracing procedure. La Mura (1999) uses a homotopy in the complex domain to find all equilibria. In some implementations the homotopy is traced explicitly on the vertices of the

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simplices in a pseudomanifold, as in Eaves (1972, 1984), Eaves and Lemke (1981), Eaves and Scarf (1975), Scarf and Hansen (1973), and the version implemented in the package *Gambit* by McKelvey et al. (1996); cf. McKelvey and McLennan (1996). In some versions the homotopy is implicit because calculations are conducted in the strategy space, as in the adaptations of the basic Lemke and Howson (1964) algorithm by Lemke (1965), Shapley (1974), and Wilson (1992) for 2-player games and its extensions by Rosenmüller (1971) and Wilson (1971) for $N$-player games, and as in the interior-point method developed by van den van den Elzen and Talman (1991). A referee noted that even McKelvey and Palfrey's (McKelvey and Palfrey, 1995) method of computing quantal-response equilibria can be interpreted as a homotopy with respect to the parameter that is the error level of quantal responses.

In previous articles (Govindan and Wilson, 2002, 2003) we describe applications of Smale's (Smale, 1976) global Newton method to computing equilibria of $N$-player games in normal form and extensive form. We show that its path is a homotopy whose image in the strategy space coincides when $N = 2$ with the path of the Lemke–Howson algorithm. The homotopy is well defined: over each generic ray through the target game, if the starting game has a unique equilibrium (and thus its index is $+1$), the corresponding path in the equilibrium graph is one dimensional and without branches or loops. After reaching the target game, the path finds a sequence of equilibria of the target game whose indices alternate between $+1$ and $-1$ if the game is generic. Our purpose in this article is to improve the performance of the global Newton method by providing a 'fast start' that jumps quickly to the vicinity of the target game. The next sections suggest the motivation for our approach.

### 1.1. Motivation

For games with more than two players, homotopy methods follow paths in the strategy space that are nonlinear. Traversing nonlinear paths requires many small steps, or larger steps require error–correction procedures, or pseudomanifold representations require successive refinements to obtain reasonable accuracy. The computational burden is compounded severely, moreover, by the characteristic feature that the path in the strategy space over the ray from the starting game to the target game has many twists and reversals. That is, the path typically involves many changes in the support of the strategies, and further, the trajectory reverses orientation where the index changes sign. A generic game along the path can have a number of equilibria that is exponential in the number of strategies (von Stengel, 1999), and the path can wind back and forth through many equilibria before making progress toward the target game. Even for games with only five players and five pure strategies per player, it is not unusual to see homotopy-based algorithms idle for many iterations at each of several intermediate games over which the trajectory oscillates back and forth through many equilibria (30 in some examples) with alternating indices $+1$ and $-1$ before resuming forward progress. This is consistent with Kohlberg and Mertens' (Kohlberg and Mertens, 1986, Theorem 1) structure theorem, which says that the Nash graph is homeomorphic to the space of games, but it is discouraging to discover that the number of folds of the graph above a typical game – not the target game, just an intermediate game on the