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On the variational equilibrium as a refinement of the generalized Nash equilibrium[☆]

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ABSTRACT

We are concerned with a class of games in which the players' strategy sets are coupled by a shared constraint. A widely employed solution concept for such *generalized* Nash games is the generalized Nash equilibrium (GNE). The variational equilibrium (VE) (Facchinei & Kanzow, 2007) is a specific kind of GNE characterized by the solution of the variational inequality formed from the common constraint and the mapping of the gradients of player objectives. Our contribution is a theory that provides sufficient conditions for ensuring that the existence of a GNE implies the existence of a VE; in such an instance, the VE is said to be a *refinement* of the GNE. For certain games, these conditions are shown to be necessary. This theory rests on a result showing the equality of the Brouwer degree of two suitably defined functions, whose zeros are the GNE and VE, respectively. This theory has a natural extension to the primal–dual space of strategies and Lagrange multipliers corresponding to the shared constraint. Our results unify some known results pertaining to such equilibria and provide mathematical substantiation for ideas that were known to be appealing to economic intuition.

1. Introduction

This paper concerns noncooperative N-player *generalized Nash games* (Harker, 1991) (or *coupled constrained games* (Rosen, 1965)) where players are assumed to have continuous strategy sets that are *dependent* on the strategies of their adversaries. Such games represent generalizations of classical noncooperative games that have traditionally allowed for strategic interactions between players to be expressed only through their objective functions. In a frequently encountered class of generalized Nash games, player strategies are required to satisfy a *common* coupling constraint. These games are called generalized Nash games with *shared constraints* (Facchinei & Kanzow, 2007; Rosen, 1965) and they form the focus of this paper.

Let $\mathcal{N} = \{1, 2, \ldots, N\}$ be a set of players, $m_1, \ldots, m_N$ be positive integers and $m = \sum_{i=1}^{N} m_i$. For each $i \in \mathcal{N}$, let $U_i \subseteq \mathbb{R}^{m_i}$ represent player i's strategy set, $x_i \in U_i$ be his strategy and $\varphi_i : \mathbb{R}^m \to \mathbb{R}$ be his objective function. We use the following notation: by x we denote the tuple $(x_1, x_2, \ldots, x_N)$, x^{-i} denotes the tuple $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N)$ and (y_i, x^{-i}) the tuple $(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_N)$. A *shared constraint* is a requirement that the tuple x be constrained to lie in a set $\mathbb{C} \subseteq \mathbb{R}^m$. In the generalized Nash game with shared constraint $\mathbb{C}$, player i is assumed to solve the parameterized optimization problem,

$$A_i(x^{-i}) \quad \underset{x_i}{\text{minimize}} \quad \varphi_i(x_i; x^{-i}) \qquad \text{subject to} \quad x_i \in K_i(x^{-i}).$$

For each $i \in \mathcal{N}$, the set-valued maps $K_i : \prod_{j \neq i} \mathbb{R}^{m_j} \to 2^{\mathbb{R}^{m_i}}$ and the map $K : \mathbb{R}^m \to 2^{\mathbb{R}^m}$, are defined as

$$K_i(x^{-i}) := \{y_i \in \mathbb{R}^{m_i} \mid (y_i, x^{-i}) \in \mathbb{C}\}, \quad \forall i \in \mathcal{N}$$
$$\text{and} \quad K(x) := \prod_{i \in \mathcal{N}} K_i(x^{-i}) \quad \forall\, x \in \mathbb{R}^m \tag{1}$$

where the notation 2^X stands for the set of all subsets of a set X. For simplicity, we have dropped the sets U_i in the above optimization problems and have assumed that $\mathbb{C}$ is contained in $\prod_{i \in \mathcal{N}} U_i$.

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Throughout this paper, we denote the game resulting from the above optimization problems by $\mathcal{G}$. The solution concept applied to analyze such games is called the *generalized Nash equilibrium* (GNE).

Definition 1.1 (*Generalized Nash Equilibrium*)**.** A strategy tuple $x \equiv (x_1, x_2, \ldots, x_N)$ is a generalized Nash equilibrium of $\mathcal{G}$ if $x_i \in SOL(A_i(x^{-i}))$ for all $i \in \mathcal{N}$.

Here SOL(P) refers to the solution set of an optimization problem P. The GNE is a special case of the *social equilibrium* proposed by Debreu (1952) for the case of general coupling constraints; see also Rosen (1965), Arrow and Debreu (1954) and the recent survey (Facchinei & Kanzow, 2007) for more on this. We now introduce another solution concept: the variational equilibrium (VE) which is a specific kind of GNE defined in Facchinei and Kanzow (2007), Facchinei and Pang (2009):

Definition 1.2 (*Variational equilibrium (VE)*)**.** A strategy tuple x is said to be a variational equilibrium of $\mathcal{G}$ if x is a solution of (VI($\mathbb{C}$, F)).

The notation (VI($\mathbb{C}$, F)) denotes a *variational inequality* with mapping F and a set $\mathbb{C}$ (see Section 1.1), where $F\colon \mathbb{R}^m \to \mathbb{R}^m$ is the function given by

$$F(x) = (\nabla_{x_1}\varphi_1(x), \ldots, \nabla_{x_N \varphi_N(x)}) \quad \forall\, x \in \mathbb{R}^m, \tag{2}$$

where ∇_{x_i} (henceforth abbreviated as ∇_i) denotes the partial derivative with respect to x_i.

The goal of this paper is to provide a theory that gives sufficient conditions for the VE to be a *refinement* of the GNE. A refinement of the set of equilibria of a game is (a) a *subset* satisfying a certain *rule*, where this rule has the property that (b) any game with a nonempty set of equilibria also possesses an equilibrium satisfying this rule. Both the refined equilibria and the generating rule are collectively referred to as the refinement. From an economic standpoint, the notion of the refinement of an equilibrium is rooted in the belief that the concept of this equilibrium may be far too weak to serve as a solution concept. For instance, if the weakness of the original concept is on the count that certain equilibria have less economic justification, then a refinement should formalize this by excluding such equilibria. Refinements of equilibria have been previously sought in both static and dynamic games: trembling hand perfect (Selten, 1975) and proper (Myerson, 1978) equilibria are refinements of mixed Nash equilibria in static finite strategy games (Başar & Olsder, 1999; Myerson, 1997; Weibull, 1997); the subgame-perfect Nash equilibrium is a refinement of the Nash equilibrium of a dynamic game (see Nisan, Roughgarden, Tardos, and Vazirani (2007, ch. 3.8)). It is known from Facchinei, Fischer, and Piccialli (2007) that every VE is a GNE. Thus this paper focuses on showing that, under suitable conditions, the existence of a GNE implies the existence of a VE.

GNEs of games such as $\mathcal{G}$ have properties that, we believe, warrant a refinement. These games are known to admit a large number, and in some cases, a manifold of GNEs (see Facchinei and Kanzow (2007); also Theorem 16 in Appendix A.1). In fact, in the following example, *every* strategy tuple in $\mathbb{C}$ is a GNE.

Example 1 (*Game Where Every Strategy Tuple is a GNE*)**.** Consider a game where player i has real valued strategies and solves

$$A_i(x^{-i}) \quad \underset{x_i}{\text{minimize}} \quad x_i \ell(X) \qquad \text{subject to} \quad X = \alpha : \lambda_i,$$

where $X = \sum_{i \in \mathcal{N}} x_i$ for $x_i \in \mathbb{R}$ for each $i \in \mathcal{N}$ and λ_i is the Lagrange multiplier for the constraint $X = \alpha$ for player i. Such games arise commonly in network routing problems. The Karush–Kuhn–Tucker (KKT) conditions characterizing the GNE, x^*, of this game are given by

$$(x_i^* \ell(X^*))' = \lambda_i, \quad \forall i \in \mathcal{N} \text{ and } X^* = \alpha.$$

Clearly, every point in the set $\mathbb{C} = \{x \mid X = \alpha\}$ is a GNE of this game. Does a subset of these characterize economically justifiable strategic behavior? □

Another shortcoming of the GNE is that there are settings in which not every GNE is meaningful from a real-world standpoint. This shortcoming provides the *first* motivation for our study which is to present a refinement of the GNE that will retain a set of GNEs that is smaller, yet economically meaningful, even under these settings. It may be argued that the VE does indeed possess such a property. Consider a game similar to that in the above example where the Lagrange multipliers corresponding to the shared constraints can be interpreted as *prices* charged on the players by an administrator for whom the players are anonymous. The VE is also known to be the GNE with equal Lagrange multipliers corresponding to the shared constraint (Facchinei et al., 2007). Thus for this game the VE has the additional property of being an equilibrium with *uniform prices* whereas the GNE corresponds to one with discriminatory prices. Since players are anonymous and hence indistinguishable from each other, it is unreasonable to assume that the administrator can charge discriminatory prices and the only equilibria that make sense are those where the same price is charged to all players, i.e. the VE.

Our *second* motivation arises from the need to characterize and compute GNEs. In general, obtaining a GNE requires a solution of an ill-posed system which leads to a *quasi-variational inequality* in the primal-space and a non-square complementarity problem in the primal–dual space. The VE, on the other hand, requires the solution of either a variational inequality (primal space) or a square complementarity problem (primal–dual space) both of which are far more tractable objects. To demonstrate this, consider the game $\mathcal{G}$ in which the set $\mathbb{C} = \{x \mid c(x) \geq 0, x \geq 0\}$ for a continuously differentiable concave function $c\colon \mathbb{R}^m \to \mathbb{R}^n$. Assuming that an appropriate constraint qualification holds (Facchinei & Pang, 2003), a vector x is a GNE of $\mathcal{G}$ if the KKT conditions for optimality of x_i for problem $A_i(x^{-i})$ hold for each player $i \in \mathcal{N}$, i.e., for each $i \in \mathcal{N}$, x_i satisfies

$$\begin{aligned} &0 \leq x_i \perp \nabla_i \varphi_i(x) - \nabla_i c(x)^T \lambda_i \geq 0 \\ &0 \leq \lambda_i \perp c(x) \geq 0, \end{aligned} \tag{KKT$_i$}$$

for some Lagrange multipliers $\lambda_i \in \mathbb{R}^n$ corresponding to the constraint $c(\cdot) \geq 0$. Note that λ_i is a vector in $\mathbb{R}^n$ and the index i corresponds to player i. For $u, v \in \mathbb{R}^n$, the notation $0 \leq u \perp v \geq 0$ means $u, v \geq 0$ and $u_j v_j = 0$ for $j = 1, \ldots, n$. In the system $\{\text{KKT}_1, \ldots, \text{KKT}_N\}$, each vector λ_i, $i \in \mathcal{N}$ is orthogonal to the same requirement "$c(x) \geq 0$", suggesting that the system is ill-posed. On the other hand, a VE is a strategy tuple x which satisfies the above system of equations for some $\lambda := \lambda_1 = \cdots = \lambda_N$, thereby resulting a square well-posed complementarity problem. Consequently, it has been common practice (Facchinei & Kanzow, 2007; Leyffer & Munson, 2005; Pang & Fukushima, 2005) to compute the VE instead of the GNE.

Let $\mathcal{S}$ be the set of games $\mathcal{G}$ for which the VE is a refinement of the GNE and suppose the subset of games for which the VE exists is denoted by $\mathcal{S}_2$. Traditional sufficiency conditions for the solvability of variational inequalities (such as those in Facchinei and Pang (2003)) do not exploit the existence of a GNE to show a solution to (VI($\mathbb{C}$, F)), thereby limiting computation to only those cases where the existence of a VE can be claimed *independently* of knowledge of the existence of a GNE. We refer to this class as $\mathcal{S}_2'$ and it is a subset of $\mathcal{S}_2$.