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Essential components of Nash equilibria for games parametrized by payoffs and strategies☆

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ABSTRACT

We study the stability of Nash equilibria for games in normal form, parameterized by continuous and quasi-concave payoffs and compact convex strategy sets in Euclidean spaces. And we establish an existence theorem for essential connected components of Nash equilibria, which includes corresponding results in the literature as special cases.

1. Introduction and preliminaries

The selection of Nash equilibria has always been a focus both in theory and applications in noncooperative game theory. In order to deal with such a problem, many concepts of refinement of Nash equilibria have been proposed since 1960s. Among them, for instance, are subgame perfect equilibrium and perfect equilibrium by Selten [1,2], proper equilibrium by Myerson [3], sequential equilibrium by Kreps and Wilson [4] and essential equilibrium by Wu and Jiang [5], etc. A comprehensive survey on this topic was given by van Damme [6]. Unfortunately, as Kohlberg and Mertens [7] argued, all the concepts of single-valued refinements mentioned above and many others fail to satisfy all or even most plausible requirements, hence a reasonable solution should be set-wise. They showed that for any finite game, there are finite connected components of Nash equilibria and at least one of them is essential in the sense that it is robust against the perturbations of payoffs. Actually, in 1963 Jiang [8] also introduced the notion of essential components of Nash equilibria for finite games and proved an existence theorem by applying set-valued analysis. Hillas [9] introduced another version of set-wise stability by considering robustness of equilibria against perturbations of best responses of the games. Although Hillas' notions satisfy the plausible requirements, it is an indirect way since there is no natural way to measure the closeness of games by corresponding best responses.

Generally speaking, most of the refinements of Nash equilibria in the literature involve some continuity. Specifically, they involve some robustness of equilibria against perturbations of either payoff functions and/or strategy spaces. For instance, perfect equilibrium and proper equilibrium are robust against some perturbations of mixed strategies. Essential equilibrium [5] and essential component of Nash equilibria [8,7] are robust against the perturbations of payoff functions. Technically, one usually requires some continuity and convexity conditions for payoff functions and strategy spaces. For finite games, strategy sets are simplexes in Euclidean spaces and payoffs usually are expected utilities which are multi-linear functions on the strategy spaces. Recently, by applying Ky Fan's inequality approach [10], Yu and Xiang [11] investigated the stability of Nash equilibria and showed that every noncooperative game (satisfying some continuity and convexity conditions) possesses at least one essential connected component of its Nash equilibria, which is also robust against the perturbations of payoffs. Yang and Yu [12] introduced the notion of essential connected components of weakly Pareto–Nash equilibria and obtained an existence theorem of essential components of weakly Pareto–Nash equilibria for multiobjective

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noncooperative games. In their models, the payoff functions satisfy some weaker continuity but are concave with respect to the strategy of the corresponding player. Unfortunately, concavity is still a rather stronger condition for applications.

The aim of this paper is to investigate essential connected components of Nash equilibria intensively. As is well known, by applying Brouwer's fixed point theorem, Nash [13] gave a map (named Nash map) to prove the existence of equilibria for finite games. Recently, for a game with convex compact strategy sets in Euclidean spaces and continuous and quasi-concave payoffs, Becker and Chakrabarti [14] construct a continuous map from the product of strategy spaces to itself and show that the fixed points are Nash equilibria of the game. In this paper, we first establish an existence theorem of essential components of fixed points. Then, by applying Becker and Chakrabarti's map we establish the existence theorem of essential connected components for noncooperative games with continuous and quasi-concave payoff functions and convex compact strategy spaces in Euclidean spaces, under perturbations of both payoffs and strategies.

First we recall some notions. For any set A, let 2^A denote the collection of all nonempty subsets of A. Let (X, d) and (Y, ρ) be two metric spaces and $F : Y \to 2^X$ be a set-valued map. Then (1) F is said to be upper semicontinuous at $y \in Y$ if for any open set O of X with $O \supset F(y)$, there exists a neighborhood $U(y)$ of y such that for each $y' \in U(y)$, $F(y') \subset O$; (2) F is said to be upper semicontinuous on Y if F is upper semicontinuous at each point $y \in Y$; and (3) F is said to be an usco map if F is upper semicontinuous on Y and $F(y)$ is compact for each $y \in Y$.

Let Y be a metric space, X be a compact metric space and $F : Y \to 2^X$ be an usco map. For each $u \in Y$, the connected component of a point $x \in F(u)$ is the union of all connected subsets of $F(u)$ containing x. Note that (see p. 356 in [15]) connected components are connected closed subsets of X and are thus connected compact. It is easy to see that the connected components of two distinct points of $F(u)$ either coincide or are disjoint, so that all connected components constitute a decomposition of $F(u)$ into connected pairwise disjoint compact subsets, i.e.,

$$F(u) = \bigcup_{\alpha \in \Lambda} C_\alpha(u),$$

where Λ is an index set, for any $\alpha \in \Lambda$, $C_\alpha(u)$ is nonempty, connected and compact and for any $\alpha, \beta \in \Lambda(\alpha \neq \beta)$, $C_\alpha(u) \cap C_\beta(u) = \emptyset$.

For each $u \in Y$, let $e(u)$ be a nonempty closed subset of $F(u)$, $e(u)$ is called an essential set of $F(u)$ if for any open set O of X with $O \supset e(u)$, there exists $\delta > 0$ such that for any $u' \in Y$ with $\rho(u, u') < \delta$, $F(u') \cap O \neq \emptyset$. An essential set $m(u)$ of $F(u)$ is said to be minimal if it is a minimal element in the family of all essential sets of $F(u)$ ordered by set inclusion. If a component $C_\alpha(u)$ of $F(u)$ is essential, then $C_\alpha(u)$ is called an essential component of $F(u)$.

Here we give the interpretation for the notions above. When we consider the stability of the solutions of some problems defined on the underlying space X (usually a metric space), Y usually consists of some problems (satisfying some conditions) endowed with some topology (usually a metric). And F denotes the solution map from problem space Y to underlying space X. The stability of the solutions then usually means some continuity of the map F.

We need the following lemmas:

Lemma 1.1 (*Lemma 3.3 in [12]*). *Let X, Y and Z be three metric spaces, $F : Y \to 2^X$ be an usco map and $G : Z \to 2^X$ be a set-valued map. Suppose that there exists a continuous map $T : Z \to Y$ such that $G(v) \supset F(u) = F(T(v))$ for each $v \in Z$. Suppose furthermore that there exists at least one essential component of $F(u)$ for each $u \in Y$. Then there exists at least one essential component of $G(v)$ for each $v \in Z$, where $v = T(u)$.*

The following lemma is a special case of Theorem 2.1 in [16].

Lemma 1.2. *Let $A_k, A (k = 1, 2, \ldots)$ be compact subsets in a metric space E and $x_k \in A_k$, $k = 1, 2, \ldots$. If $h(A_k, A) \to 0$, where h is the Hausdorff metric on E, then $\{x_k\}$ has a subsequence converging to an element in A.*

Lemma 1.3. *Let A, A', A'' be compact convex subsets in a Euclidean space E, satisfying $A' \subset int A$, $A'' \subset int A$, where $int A$ is the relative interior of A in E. If $h(A, A'') < d(\partial A, A')$, then $A' \subset A''$, where h is the Hausdorff metric on E, ∂A is the boundary of A and $d(\partial A, A') = \min_{x \in \partial A, y \in A'} d(x, y)$.*

Proof. By way of contradiction, suppose that $A' \not\subset A''$. Then there is an $x' \in A'$ such that $x' \notin A''$. By variational lemma, there is a unique $x'' \in A''$ such that $\|x' - x''\| = d(x', A'')$. Since A, A', A'' are compact and $A' \subset int A$, $A'' \subset int A$, we have $d(\partial A, A') > 0$ and $h(A, A'') = \max_{z \in \partial A} d(z, A'')$. Since $x', x'' \in int A$ and A is compact and convex, there is an $x \in \partial A$ such that x' lies between x'' and x in a line. Clearly, $h(A, A'') \geq h(x, A'') = \|x - x''\| \geq \|x' - x''\| = d(\partial A, A')$, a contradiction. ■

2. The essential components of fixed points

Let X be a nonempty compact convex subset of a Euclidean space E. Let $L(X)$ be the collection of all continuous functions $f : X \to X$. Let $CK(X)$ be the collection of all nonempty compact convex subsets of X. Denote $M = \{u = (\varphi, A) \in L(X) \times CK(X) : \exists x \in A, s.t. \varphi(x) \in A\}$ and define a metric ρ on M as follows:

$$\forall u = (\varphi, A), v = (\psi, B) \in M, \quad \rho(u, v) = \max_{x \in X} \|\varphi(x) - \psi(x)\| + h(A, B),$$

where h is the Hausdorff metric on E. Given a $u = (\varphi, A) \in M$, a point $x \in X$ is called a fixed point of u in A if $x \in A$ and $x = \varphi(x)$. For any $u \in M$, denote by $F(u)$ the set of all fixed points of φ in A. By Brouwer fixed point theorem, $F(u) \neq \emptyset$ and thus $u \mapsto F(u)$ defines a set-valued map from M to X.