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Deriving Nash equilibria as the supercore for a relational system☆

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ABSTRACT

In this paper, under a binary relation that refines the standard relation which only accounts for single profitable deviations, we obtain that the set of NE strategy profiles of every finite non-cooperative game in normal form coincides with the supercore (Roth, 1976) of its associated abstract system. Further, under the standard relation we show when these two solution concepts coincide.

1. Introduction

In game theory there has been some interest in the study of the equivalence between solution concepts for non-cooperative games in normal form and those for the abstract system (*i.e.*, an abstract set endowed with a binary relation) associated with them.[1] These analyses have focused on defining a binary relation for which the solution concepts under study coincide.[2] This is important because finding such a binary relation contributes to improving our understanding of how solution concepts in game theory relate to each other.

For instance, Kalai and Schmeidler (1977) associate the mixed extension of a normal form game with an abstract system using a binary relation that only accounts for profitable single deviations. For this binary relation, they find no equivalence between the Nash equilibrium (NE) solution (Nash, 1951) and the admissible set (Kalai et al., 1976). The coincidence, however, is achieved under a somewhat different binary relation that incorporates the idea of rationalizability. Greenberg (1989), and Kahn and Mookherjee (1992) for the case of infinite games, show that the Coalition Proof Nash Equilibrium solution is

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[1] Abstract systems are considered by von Neumann and Morgenstern in their book Theory of Games and Economic Behavior, 1947). A generalization of this notion, the general system, has been defined by Luo (2001).

[2] Greenberg (1990) considers different binary relations that associate games in normal form with abstract relational systems.

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equivalent to the (von Neumann and Morgenstern (vN&M)) stable sets solution under a binary relation that allows for coalitional deviations.

More recently, Inarra et al. (2007) study the supercore (Roth, 1976) for the abstract system associated with a finite game in normal form by considering the binary relation used by Kalai and Schmeidler (1977). For the case of pure strategies, a sequence of games is given and it is shown that the supercore for the abstract system associated with the first game in the sequence coincides with the set of NE strategy profiles of the last game in that sequence. With regard to the mixed extension of the game, it is shown that the set of NE strategy profiles coincides with the supercore for games with a finite number of NE.

Along this line of research, the purpose of this paper is to define a suitable binary relation under which we may obtain that the set of NE strategy profiles for the mixed extension of *every* finite non-cooperative game in normal form coincides with the supercore for the abstract system associated with it. The new relation that we propose simply refines the conventional one Kalai and Schmeidler (1977) by incorporating individual deviations that do not require a strictly positive gain.

The contributions of the paper may be summarized as follows. First, using the new binary relation we obtain that the set of NE strategy profiles coincides with the supercore for *every* finite game. Second, using numerical examples we show that under the standard relation this coincidence need not hold for games with *infinite* NE profiles.[3] Lastly, we establish that the two solution concepts under study coincide *if and only if* there is not a *non-NE strategy profile* in which a player's payoff is equal to her payoff in some NE strategy profile and that is *only* dominated by "some strategy profile which is dominated by some NE strategy profile."

The rest of the paper is organized as follows. Section 2 contains the preliminaries. In Section 3 we give some examples to provide some intuition for the upcoming results. Section 4 contains the definition of the new binary relation and the results.

2. Preliminaries

An *abstract system* is a pair (X, R), where X is a set of elements and R is an irreflexive binary relation, which may be partial, defined on X. The relation R reads "dominates." Hence, if for two elements x, x' in X we have xRx', then we say that x dominates x'.

For any $x \in X$, let $\mathcal{D}(x)$ denote the dominion of x, *i.e.*, $\mathcal{D}(x) = \{x' \in X : xRx'\}$. Given a non-empty subset A of X, we may define the following sets: $\mathcal{D}(A) = \bigcup_{x \in A} \mathcal{D}(x)$ is the set of elements dominated by some element of A and $\mathcal{U}(A) = X - \mathcal{D}(A)$ is the set of elements undominated by any element of A.

A set $A \subseteq X$ is the *core* for (X, R) if $A = \mathcal{U}(X)$.

A set $A \subset X$ is a *vN&M stable set* of (X, R) if $A = \mathcal{U}(A)$. Hence $A \subset \mathcal{U}(A)$, which is known as the internal stability condition, and $\mathcal{U}(A) \subset A$, known as the external stability condition.

A *subsolution* of (X, R) is a subset A of X such that *(i)* $A \subset \mathcal{U}(A)$ and *(ii)* $A = \mathcal{U}^2(A)$, where $\mathcal{U}^2(A) = \mathcal{U}(\mathcal{U}(A))$.

Let $\mathcal{P}(A) = \mathcal{U}(A) - A$ be the set of elements undominated by any element of A excluding the elements of A. Given a subsolution A, the set X may be partitioned into three sets: A, $\mathcal{D}(A)$ and $\mathcal{P}(A)$.[4] Moreover, if $A \subset \mathcal{U}(A)$, given that $\mathcal{U}(\mathcal{U}(A)) = \mathcal{U}(A \cup \mathcal{P}(A)) = X - \mathcal{D}(A \cup \mathcal{P}(A)) = A \cup \mathcal{P}(A) - \mathcal{D}(\mathcal{P}(A))$, then we have $\mathcal{U}^2(A) = A \Leftrightarrow \mathcal{P}(A) \subset \mathcal{D}(\mathcal{P}(A))$ and $A \cap \mathcal{D}(\mathcal{P}(A)) = \varnothing$.

The most significant subsolution of (X, R) is the intersection of all subsolutions which is called the *supercore*.[5]

This solution concept is a generalization of the vN&M stable set. Note that if A is a vN&M stable set then A is a subsolution with $\mathcal{P}(A) = \varnothing$.

A finite *normal form game* Γ^N is a triple $< N, \left\{S_i\right\}_{i \in N}, \left\{u_i\right\}_{i \in N} >$ where $N = \{1, \ldots, n\}$ is the finite set of players, S_i is the finite set of strategies for player i and $u_i : S = \times_{i \in N} S_i \to \mathbb{R}$ is player i's payoff function.

A *mixed extension of the game* Γ^N is a triple $< N, \{\Delta S_i\}_{i \in N}, \left\{U_i\right\}_{i \in N} >$ where ΔS_i is the simplex of the mixed strategies for player i, and $U_i : \Delta(S) = \times_{i \in N} \Delta(S_i) \to \mathbb{R}$, assigns to $\sigma \in \Delta(S)$, where σ denotes a mixed strategy profile, the expected value under u_i of the lottery over S that is induced by σ, so that $U_i(\sigma) = \sum_{s \in S} (\prod_{j \in N} \sigma_j(s_j) u_i(s))$.

The strategy profile $\sigma^* = (\sigma_1^*, \ldots, \sigma_n^*)$ is a *Nash equilibrium* in the mixed extension of the game Γ^N if σ_i^* is a best response to $\sigma_{-i}^* = (\sigma_1^*, \ldots, \sigma_{i-1}^*, \sigma_{i+1}^*, \ldots, \sigma_n^*)$ for all $i \in N$. The set of NE strategy profiles of a *mixed extension of the game* Γ^N is denoted by Σ^*.

In order to associate an abstract system (X, R) to the mixed extension of a finite normal form game we follow the approach developed by Greenberg (1990). He proposes a negotiation procedure among players that can be described as follows.[6] Suppose that the strategy profile σ is proposed to players. Then each individual player can object to the prevailing profile and can threat the others by saying that she will choose another strategy. If she does this, player i induces σ' from σ. The set

[3] Inarra et al. (2007) proves the coincidence result for *finite* games with a finite number of NE profiles. Wilson (1971) shows that in "almost all" finite games the number of NE is finite and odd. See also Harsanyi (1973).

[4] These sets are the good, the ugly and the bad in terms of Kahn and Mookherjee (1992).

[5] If $\mathcal{U}(X) = \varnothing$, then the supercore of (X, R) is the empty set (Roth, 1976).

[6] In Greenberg (1990) this procedure is called an individual contingent threat situation.