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On the evolutionary selection of sets of Nash equilibria

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Abstract

It is well established for evolutionary dynamics in asymmetric games that a pure strategy combination is asymptotically stable if and only if it is a strict Nash equilibrium. We use an extension of the notion of a strict Nash equilibrium to sets of strategy combinations called 'strict equilibrium set' and show the following. For a large class of evolutionary dynamics, including all monotone regular selection dynamics, every asymptotically stable set of rest points that contains a pure strategy combination in each of its connected components is a strict equilibrium set. A converse statement holds for two-person games, for convex sets and for the standard replicator dynamic.

JEL classification: C72; C79

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1. Introduction

Under a wide range of evolutionary dynamics, a pure-strategy Nash equilibrium of an asymmetric normal-form game will be asymptotically stable if and only if it is a strict Nash equilibrium. [1] This observation has been made repeatedly, for instance by Eshel and Akin [12], Samuelson and Zhang [30] and Ritzberger and Weibull [29], see also Ritzberger and Vogelsberger [28].

The objective of this paper is to capture in a similar vein the dynamic stability properties of sets of Nash equilibria.

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[1] Loosely speaking, a strategy combination is asymptotically stable if all trajectories starting nearby stay close and converge to it in the long run.

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Many games have Nash equilibrium components that do not consist of isolated equilibria. If a strategy combination is contained in such a component then it will not be asymptotically stable because Nash equilibria are rest points. Accordingly, we will investigate which Nash equilibrium components or, more generally, which sets of Nash equilibria can be asymptotically stable under a wide range of evolutionary dynamics.

The sets of Nash equilibria that play an important role in our paper are called strict equilibrium sets. Strict equilibrium sets generalize strict Nash equilibria to sets of strategy combinations. Following Balkenborg [3], a set of Nash equilibria is called a *strict equilibrium set* (short, SE set) if for any element in the set, either a player loses strictly by deviating unilaterally or else his deviation leads to another Nash equilibrium in the set. For instance, the set of strategy combinations leading to the forward induction outcome for a twice repeated battle-of-the-sexes game (see van Damme [38]) is a SE set. Further examples of SE sets are provided throughout the paper.

We first study the asymptotic stability of sets of Nash equilibria under the standard replicator dynamic (Taylor [36]). Here we obtain a very clear-cut characterization. A set of Nash equilibria is an asymptotically stable set if and only if it is a SE set. The proof of asymptotic stability relies on an alternative characterization of SE sets that is motivated by Thomas' [37] notion of an evolutionarily stable set.

However, while some theoretical results single out the standard replicator dynamic as a central learning and imitation dynamic (e.g. [8,14,32]), the standard replicator dynamic is a knife-edge case in the sense that very similar dynamics may behave very differently. We are hence interested in extending our analysis to a broad class of evolutionary dynamics that share only some basic qualitative features with the standard replicator dynamic.

The more general evolutionary dynamics we consider are dynamics that "reinforce best replies". By this we mean that the dynamic is a regular selection dynamic—as defined in Samuelson and Zhang [30]—and that it satisfies the list of intuitive properties discussed in Section 4, which link growth rates and payoffs. For instance, best replies must have non-negative growth rates. Our definition encompasses all sign preserving [26] and monotone dynamics [30].

Given our results on the standard replicator dynamic, only SE sets can be asymptotically stable under *all* dynamics that reinforce best replies. However, other sets of Nash equilibria can be asymptotically stable under *some* dynamics that reinforce best replies. For instance, it is well known that the interior mixed strategy Nash equilibrium in Matching Pennies is asymptotically stable under the adjusted replicator dynamic (see, e.g. Maynard Smith [22, Appendix J]). Therefore, we restrict attention to sets that contain a pure strategy combination in each connected component, a property shared by SE sets. We show that a set of Nash equilibria with this property is a SE set if it is asymptotically stable under some dynamic reinforcing best replies.

Finally, we consider the converse and ask whether a SE set is asymptotically stable under *any* dynamic reinforcing best replies. We can currently prove this only if the SE set satisfies one additional restriction, namely, each strategy combination in the SE set must be a best reply to all best replies against it. This condition is always satisfied for SE sets in two-player games or for SE sets that are convex.

For two-player games our results imply the following tight characterization of asymptotic stability in purely static, game theoretic terms. A set containing a pure strategy combination in each connected component is an asymptotically stable set of rest points for a given dynamic reinforcing best replies if and only if it is a SE set.

In a related study, Ritzberger and Weibull [29] show for a slightly different class of evolutionary dynamics that a face is asymptotically stable if and only if it is *closed under better replies*. (There is no condition on how the dynamic behaves within the set.) To be closed under better replies means