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Pseudocontinuous functions and existence of Nash equilibria

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Abstract

In topological spaces, we introduce a new class of functions (pseudocontinuous functions) and we present some characterizations and properties. In particular, we show that any preference relation endowed of utility functions is continuous if and only if any utility is pseudocontinuous. A maximum theorem is proved for such a class of functions and connections with similar results are investigated. Finally, the existence of Nash equilibria for games with pseudocontinuous payoffs is obtained.

Keywords: Non cooperative game; Nash equilibrium; Pseudocontinuous function; Numerical representation of a preference; Maximum theorem

1. Introduction

Following the early theorems for non cooperative games (Nash, 1950, 1951; Glicksberg, 1952), the payoffs have to be continuous in order to obtain the existence of Nash equilibria. However, several games as the oligopolies of Bertrand (1883) and Hotelling (1929) have discontinuous payoffs and several authors have studied the existence of equilibria when the payoffs are not necessarily continuous. Among others, we remind the reader to Dasgupta and Maskin (1986), Vives (1990), Baye et al. (1993), Tian and Zhou (1995), Cavazzuti (1996), Reny (1999) and Lignola and Morgan (2002), where only the lower semicontinuity is relaxed or there are not explicit assumptions on any data. In this paper, we introduce a new sufficient topological condition on the payoffs (called *pseudocontinuity*) which is strictly weaker than lower semicontinuity and than

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upper semicontinuity and which is equivalent to the continuity of the associate preference relations: if a preference relation is endowed of numerical representations (also called utility functions), then the preference is *continuous* (Eilenberg, 1941; Rader, 1963; Debreu, 1964; Bergstrom, 1975) if and only if any numerical representation is a *pseudocontinuous* function. So, pseudocontinuity is the common topological property satisfied by a whichever numerical representation of a continuous preference.

The paper is organized as follows. In Section 2, (upper and lower) pseudocontinuous functions are presented together with some characterizations and properties. In particular, it is shown that every preference relation endowed of utility functions is *lower continuous* (Eilenberg, 1941; Rader, 1963; Debreu, 1964; Bergstrom, 1975) if and only if every utility is *upper pseudocontinuous*. The relationships between upper (resp. lower) pseudocontinuity and *sequential* upper (resp. lower) pseudocontinuity (Morgan and Scalzo, 2004) conclude the section. In Section 3, since the Berge's maximum theorem (Berge, 1959) plays a crucial role in the classical proofs of the existence of Nash equilibria, a maximum theorem is obtained for pseudocontinuous functions; an example shows that it is not possible to improve further on the result by using explicit assumptions on any data. Then, a new existence result for Nash equilibria is presented in games with pseudocontinuous payoff functions. Finally, Section 2 investigates the connections between games having pseudocontinuous payoff functions and others classes of games already considered in literature.

2. Pseudocontinuous functions

In this section, we introduce the class of pseudocontinuous functions together with some characterizations and properties.

Definition 2.1. Let Z be a topological space and f be an extended real valued function defined on Z.

- f is said to be *upper pseudocontinuous at* $z_o \in Z$ if for all $z \in Z$ such that $f(z_o) < f(z)$, we have:

  $$\limsup_{y \to z_o} f(y) < f(z);$$

  f is said to be *upper pseudocontinuous on* Z if it is upper pseudocontinuous at z_o, for all $z_o \in Z$;
- f is said to be *lower pseudocontinuous at* $z_o \in Z$ if $-f$ is upper pseudocontinuous at z_o and f is said to be *lower pseudocontinuous on* Z if it is lower pseudocontinuous at z_o, for all $z_o \in Z$;
- f is said to be *pseudocontinuous* if it is both upper and lower pseudocontinuous.

Any upper (resp. lower) semicontinuous function is also upper (resp. lower) pseudocontinuous. The converse is not true. In fact, in Example 4.1 are presented pseudocontinuous functions which are neither upper nor lower semicontinuous. Moreover, we note that the class of upper (resp. lower) pseudocontinuous functions is strictly included in the class of transfer upper (resp. lower) continuous functions introduced by Tian and Zhou (1995) (see Example 3.1).

Upper pseudocontinuity is an ordinal property, as shown in the next proposition, where some characterizations of upper pseudocontinuity are also given. Here, we say that a property $\mathcal{P}$ satisfied by an extended real valued function f is *ordinal* if $\phi \circ f$ satisfies $\mathcal{P}$, for every strictly increasing function $\phi : \mathbb{R} \longrightarrow \mathbb{R}$, where $(\phi \circ f)(z) = \phi[f(z)]$ for all z such that $f(z) \in \mathbb{R}$. Moreover, in the following proposition we also show that an upper pseudocontinuous function defined on a com-