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An insurance network: Nash equilibrium

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Abstract

We consider d insurance companies whose surplus processes are r.c.l.l. functions (like the sample paths of perturbed Levy processes). Suppose they have a treaty to diversify risk; accordingly, if one company needs a certain amount to prevent ruin, the other companies pitch in previously – agreed – upon fractions of the amount, and any shortfall is got from external sources. With each company trying to minimise its repayment liability, the situation is viewed upon as a d-person dynamic game with state space constraints and a Nash equilibrium is sought. Under certain natural conditions, it is shown that the Skorokhod problem of probability theory provides a (unique) Nash equilibrium. The thrust of the paper is entirely deterministic.

Keywords: d-Person dynamic game; State space constraints; Deterministic Skorokhod problem; Orthant; Half space; Control; Vague convergence; Spectral radius; Reinsurance model; Reflection; Drift

1. Introduction

We consider d insurance companies operating under a treaty to diversify risk. Accordingly, if one company estimates at some instant of time that it needs a certain amount to prevent ruin, the other companies in the network pitch in previously – agreed – upon fractions of the amount. Any shortfall is to be obtained by the concerned company from "external" sources. The amount got from "internal" sources (viz. other members of the network) carry easier repayment terms, due to mutual obligations. The amount needed to prevent ruin is viewed upon as a control. Needless to say, each company will try to minimise its cost. The companies can possibly be in competition; and the objective of control is to keep the surplus of each company non-negative. Thus we are lead naturally to a d -person non-cooperative dynamic game with state space constraints, and we seek a Nash equilibrium. It turns out that under certain natural conditions a Nash equilibrium is provided by the solution to the appropriate deterministic Skorokhod problem in an orthant.

Skorokhod problem of probability theory has played a major role in the stochastic differential equation formulation of reflected (or regulated) Brownian motion/diffusions/Levy processes. Thanks to the impetus from queueing networks, the problem in nonsmooth domains, like a quadrant or an orthant, has attracted a lot of attention. As sample path analysis is helpful in understanding the stochastic problem, the deterministic Skorokhod problem has also been extensively studied; see Harrison and Reiman (1981), Reiman (1984), Mandelbaum and Pats (1998), Ramasubramanian (2000) and references therein. In the present work also, for similar reasons we consider the nonstochastic scenario and the game we deal with is a deterministic d-person dynamic game with state space constraints.

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The surplus/risk processes in the absence of controls are r.c.l.l. functions with time and space dependent drift. The optimally controlled process for the game is then the corresponding reflected/regulated process in an orthant; the Nash equilibrium is given by the “pushing” part of the solution to Skorokhod problem. The sample paths of perturbed Levy processes are well known examples of r.c.l.l. functions (see Rolski et al., 2001); the optimally controlled process in such a case will be the reflected Levy process (see Ramasubramanian, 2000; Dasgupta and Ramasubramanian, 2003 and the references therein). Though perturbed Levy processes and the associated reflected stochastic processes are mentioned to motivate the problem, the thrust of this paper is entirely deterministic.

In insurance mathematics, beginning perhaps with the works of Borch, game theoretic ideas have been used in the context of reinsurance by many authors; see the recent survey paper Aase (2002) and references therein. However, the flavour is quite different in our work. Besides being a continuous time model, our model has state space constraints, as the controlled process has to live in an orthant.

Optimality property of Skorokhod problem in one-dimension (in half line $[0, \infty)$) is well known; see Harrison (1985). This has also been used in Taskar (2000) in the context of dividend payment.

Game theoretic aspects of Skorokhod problem in an orthant have recently been studied in Ramasubramanian (2004), where the absolutely continuous case has been considered. Consequently the game becomes a differential game with state space constraints and the framework of HJB equations and viscosity solutions becomes available. Nevertheless it has some parallels with the present work. Incidentally optimal control problems with state space constraints were first studied by Soner, and one may see Fleming and Soner (1993), Bardi and Capuzzo-Dolcetta (1997) for accounts of this.

The paper is organized as follows. In Section 2 we present the basic network model, and the passage to the determinstic d-person game. As explained in Examples 2.1–2.4 our model with appropriate parameters can be considered as a sort of reinsurance model; the difference with conventional models being that the reinsurer helps out only when a demand is made by the cedent. Brief description of the deterministic Skorokhod problem in an orthant/a half space is given in Section 3.

In Section 4, the main body of the paper, the determinstic d-person game alluded to above and its connection to the Skorokhod problem are discussed. For this it is convenient to introduce a control problem in a half space, and show that the optimal solution is given by the solution to a Skorokhod problem in the half space. A priori bounds given in the context of Skorokhod problem help in fixing suitable compact sets where the controls lie; the topology is the topology of vague convergence of uniformly bounded measures (or sub-probability measures) on a finite interval $[0, T]$. We adopt a twin approximation procedure, one by solving a Skorokhod problem for ‘known’ function of time variable alone, and the other by solving the integral Eqs. (4.1) and (4.2). The hypotheses, which are natural in the context of Section 2, are rigged so that the cost does not increase at any stage of the approximation. Under additional assumptions, Skorokhod problem gives a unique Nash equilibrium for all $0 \leq t \leq T$.

It is well known that many parallels exist between insurance models and (one-dimensional) queueing theory; see Asmussen (2001) and the references therein. That the Skorokhod problem plays a central role in queueing networks has already been mentioned; see for example Chen and Yao (2000) and references therein. However, to our knowledge, no connection between insurance and queueing networks has been pointed out in the literature.

The question of non-ruin of the network in the continuous perturbed Brownian case is taken up in a companion paper under preparation.

Although we have used problem from insurance to elucidate the model, other situations are also meaningful; some examples are potential demand/output lost in queueing networks due to buffer being empty, feasible production plans in input-output models, and allocation of funds/subsidies to various sectors (including welfare sectors) of an interdependent economy; see various comments in Chen and Mandelbaum (1991), Ramasubramanian (2000), Reiman (1984).

Finally some comments about notation. $D([0, \infty) : E)$, $D([0, T] : E)$ denote respectively E-valued r.c.l.l. functions on $[0, \infty)$, $[0, T]$:

$$D_{\uparrow}([0, T] : \mathbb{R}_+^d) = \{y(\cdot) = (y_1(\cdot), \ldots, y_d(\cdot)) \in D([0, T] : \mathbb{R}^d) : y_i(\cdot) \geq 0,\ y_i(\cdot)\,\text{nondecreasing},\ 1 \leq i \leq d\}.$$

For $1 \leq i \leq d$, $y \in \mathbb{R}^d$ we denote $y_{-i} = (y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_d)$; for $1 \leq i \leq d$, an $\mathbb{R}^d$-valued function $y(\cdot)$, similarly $y_{-i}(\cdot) = (y_1(\cdot), \ldots, y_{i-1}(\cdot), y_{i+1}(\cdot), \ldots, y_d(\cdot))$; for real $y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_d, \xi$, we denote $(\xi, y_{-i}) = (y_1, \ldots, y_{i-1}, \xi, y_{i+1}, \ldots, y_d)$.