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Existence of Nash equilibria in large games

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ABSTRACT

Podczeck [Podczeck, K., 1997. Markets with infinitely many commodities and a continuum of agents with non-convex preferences. Economic Theory 9, 385–426] provided a mathematical formulation of the notion of "many economic agents of almost every type" and utilized this formulation as a sufficient condition for the existence of Walras equilibria in an exchange economy with a continuum of agents and an infinite dimensional commodity space. The primary objective of this article is to demonstrate that a variant of Podczeck's condition provides a sufficient condition for the existence of pure-strategy Nash equilibria in a large non-anonymous game G when defined on an atomless probability space $(T, \mathfrak{T}, \lambda)$ not necessary rich, and equipped with a common uncountable compact metric space of actions A. We also investigate to see whether the condition can be applied as well to the broader context of Bayesian equilibria and prove an analogue of Yannelis's results [Yannelis, N.C., in press. Debreu's social equilibrium theorem with asymmetric information and a continuum of agents. Economic Theory] on Debreu's social equilibrium theorem with asymmetric information and a continuum of agents.

1. Introduction

In this article, we consider a large non-anonymous game, that is to say, a measurable map G from an atomless probability space of players $(T, \mathfrak{T}, \lambda)$ to a space of players' types $\mathcal{C}$, in which each player is assumed to face an identical action space A and be completely characterized by its payoff function u alone when information asymmetry plays no role and by its payoff function u along with its private information $\mathfrak{F}$ and prior q when information asymmetry matters. We deduce that the presence of many players of every type ensures the existence of pure-strategy Nash equilibria in these games.

Rich probability spaces have been recently proven to furnish some useful regularities regarding a convexification of integrals of correspondences and a purification of measure-valued maps (Keisler and Sun, 2002b; Podczeck, 2008a, i; Sun and Yannelis, 2008).

A convexifying effect on integrals of correspondences guarantees along with other standard assumptions the existence of relevant equilibria without convexity assumptions on payoff functions in the case of games in which payoffs depend on the choices of other players only through the integral and on utility functions in the case of economies in which no externalities are present. A Lyapunov-type theorem, which provides the desired convexification, is known to hold for finite dimensional spaces provided that a measure space in question is atomless and is known not to hold generally for infinite dimensional spaces unless some stronger condition than nonatomicity are imposed on measure spaces.

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Yannelis (1991) obtained several results of the Bochner integral of a correspondence taking values in a Banach space, in particular, an infinite dimensional generalization of the work of Aumann (1965), which includes a weak compactness theorem of the Bochner integral of the closed convex hull of a correspondence, a convexity theorem of the closure of the Bochner integral of a correspondence, and an infinite dimensional version of Fatou' lemma with some conditions. Rustichini and Yannelis (1991) derived a dimensional condition which formulates the condition of many more agents than commodities and which is strong enough to warrant a convexifying effect that leads to an infinite dimensional version of Fatou's lemma and an upper semicontinuity lifting theorem. The by-products include an existence theorem of pure-strategy Nash equilibria for continuum games of Schmeidler-type with an infinite dimensional strategy space and the construction in light of Maharam's theorem of an explicit example of the measure spaces that satisfy the dimensional condition, in which the presently recognized usefulness of rich probability spaces is implicit.

Podczeck (1997) derived the condition of many players of almost every type, an alternative condition to convexify integrals of correspondences and proved the existence of Walrasian equilibria for economies with continuum of agents who trade bundles in a infinite dimensional space.

Recently Podczeck (2008a) and Sun and Yannelis (2008) demonstrated that the richness (saturation) condition, which is satisfied by atomless Loeb spaces (Hoover and Keisler, 1984) and which is weaker than Rustichini–Yannelis's dimensional condition (with the acceptance of the continuum hypothesis), suffices to guarantee a convexifying effect on integrals of correspondences in the form that is applicable to games and economies. Podczeck (2008b) also demonstrated that the richness condition can be utilized to purify measure-valued maps as in Loeb and Sun (2006), which can be applied to obtain a Nash equilibrium for finite games with incomplete information (Milgrom and Weber, 1985; Loeb and Sun, 2006).

The primary objective of this article is to demonstrate that a variant of Podczeck's condition provides a sufficient condition for the existence of pure-strategy Nash equilibria in a large non-anonymous game G when defined on an atomless probability space $(T, \mathfrak{T}, \lambda)$ not necessary rich, and equipped with a common uncountable compact metric space of actions A. We also investigate to see whether the condition of "many players of (almost) every type" can be applied as well to the broader context of Bayesian equilibria; to be more specific, whether our condition can substitute the assumption of "many more agents than strategies" in Yannelis (2008, Sections 4.2 and 5.1).

Keisler and Sun (2002a) showed that if $(T, \mathfrak{T}, \lambda)$ is any atomless Loeb probability space, $\mathcal{U}$ any Polish space, $f : T \to \mathcal{U}$ a mapping, measurable for $\mathfrak{T}$ and the Borel σ-algebra $\mathcal{B}(\mathcal{U})$, and $f_*\lambda$ denotes the image measure of λ under f on $\mathcal{U}$, then for $f_*\lambda$-almost every $u \in \mathcal{U}$ the inverse image $f^{-1}(u)$ has a cardinality at least as large as that of the continuum. In particular, when the set of players is an atomless Loeb space, then there are also "many players of almost every type". Though the condition of "many players of almost every type" in this sense may not be describable in terms of our formalization, there is no conceptual difference on the level of interpretation. Thus one contribution of this paper may be said to have shown that if one accepts the idea of "many players of almost every type" as a condition, then there is actually no need to assume Loeb spaces of agents in order to get pure-strategy Nash equilibria; instead one can work with, say [0, 1] with Lebesgue measure to specify the set of agents. (The author is indebted to an anonymous referee for informing him of the above fact about Keisler and Sun (2002a) and also for enhancing the contributions of this paper.)

We remark that there are rich measure spaces – not necessary Loeb spaces – that admit injective measurable mappings to Polish spaces, so that "many players of almost every type" is not automatically implied. The existence of such rich measure spaces was first noted in the economic literature in Podczeck (2008a, i).

It is interesting to note that Podczeck's condition requires the σ-algebra $\mathfrak{T}$ to be countably generated (modulo null sets) while the richness entails $\mathfrak{T}$ to be uncountably large, and hence our existence result covers an area the Keisler–Sun's existence result does not. We also demonstrate that as a by-product of one of the results in Keisler and Sun (2002b), one can obtain a necessary and sufficient condition for the symmetrization of a CNED in a dispersed anonymous game.

This article is organized as follows: Section 2 introduces necessary notations and definitions; Section 3 provides examples that furnish plausibility to our main assertion; Section 4 states and proves our main theorem and provides some remarks; Section 5 discusses the relationship of our work to rich probability spaces and obtains a necessary and sufficient condition for the symmetrization of a CNED in a dispersed anonymous game; Section 6 discusses large non-anonymous games with asymmetric information; and Section 7 (appendix) supplies some details for the proofs in Section 5.

2. Notations and definitions

In the sequel, $(T, \mathfrak{T}, \lambda)$ denotes a probability space where $\mathfrak{T}$ is a σ-algebra on the set T and λ a countably additive probability measure on $\mathfrak{T}$. For a topological space X, $\mathcal{B}(X)$ denotes its Borel σ-algebra and Prob(X) the set of probability measures on $\mathcal{B}(X)$. When we view Prob(X) as a topological space, we always assume that it is endowed with the weak topology. When A is a compact metric space, $\mathcal{U}_A$ denotes the set of all real-valued continuous functions on $A \times \text{Prob}(A)$ where we endow $\mathcal{U}_A$ with the topology of uniform convergence. When a probability measure ι is defined in the domain of a measurable map f, $f_*\iota$ denotes the direct image measure of ι under f.

In what follows we regard $(T, \mathfrak{T}, \lambda)$, T, for short, as a space of *players*, A as a space of possible *actions*, and $\mathcal{U}_A$ as a space of *payoff functions*.

The following definition of a game is analogous to that in Schmeidler (1973) and Khan and Sun (2002).