Nonlinear Analysis 73 (2010) 2933–2940

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Inequalities and Nash equilibria

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ARTICLE INFO

Article history:
Received 27 April 2010
Accepted 16 June 2010

Keywords:
0-pair-concavity
Generalized N-person game
Nash equilibrium
S-Nash equilibrium
Pure strategy Nash equilibrium

ABSTRACT

This paper concentrates on the problem of the existence of equilibrium points for non-cooperative generalized N-person games, N-person games of normal form and their related inequalities. We utilize the K–K–M lemma to obtain a theorem and then use it to obtain a new Fan-type inequality and minimax theorems. Various new equilibrium point theorems are derived, with the necessary and sufficient conditions and with strategy spaces with no fixed point property. Examples are given to demonstrate that these existence theorems cover areas where other existence theorems break down.

1. Introduction

In mathematical economics, the main problem for investigating different kinds of economic models is showing the existence of an equilibrium. A number of authors have proved the existence of an equilibrium in several economic models. For example, the existence of a Cournot–Nash equilibrium for a normal game was proved by Nash [1]. The notion of a generalized game (social system) was introduced by Debreu [2] which contains the normal game as a special case and then proved the existence of an equilibrium. Friedman [3] established a generalization of Nash's theorem using the quasiconcavity assumption on every payoff function. Nikaido and Isoda [4] considered a mapping of individual payoffs into an aggregate function that guarantee the existence of a Nash equilibrium. These results has been further investigated, e.g., see Refs. [5–16].

In this paper, we first introduce the 0-pair-concave condition which unifies the $\mathcal{C}$-quasiconcavity established by Hou [11], the diagonal transfer quasiconcavity (weaker than the quasiconcavity) established by Baye et al. [5], and the $\mathcal{C}$-concavity (weaker than concavity) established by Kim and Lee [7]. We then utilize the K–K–M lemma to obtain an inequality and apply this to obtain a new Fan-type inequality and a minimax theorem. After defining the aggregate payoff function and deriving a key Lemma, we have been able to establish new equilibrium theorems such as Nash, S-Nash, pure-strategy Nash, and J-dominant-strategy Nash equilibrium theorems for generalized games or normal games. These have been established with the necessary and sufficient conditions and with topological strategy spaces that do not have the fixed point property. Examples are given to demonstrate that these existence theorems cover areas in which other existence theorems break down. In several ways, our theorems generalize the corresponding results of Hou [10], Kim and Lee [9], Kim and Kum [8], and Baye et al. [5].

2. Preliminaries

Throughout this paper, all topological spaces are assumed to be Hausdorff.

2^A denotes the sets of all subsets of A. Let A be a subset of a topological space X. We denote by $cl_X A$ the closure of A in X. Let $\triangle_n$ be the standard n-dimensional simplex in R^{n+1}. If A is a subset of a vector space, we denote this by coA the convex hull of A. If $B \subset A$ and $f : A \to R$, we denote this by $f|_B$ the restriction mapping of f on B.

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0362-546X/$ – see front matter
doi:10.1016/j.na.2010.06.054

Let $I = \{1, \ldots, n\}$ be a set of players. A non-cooperative generalized N-person game is an ordered $3n$-tuple $\Gamma = \{X_1, \ldots, X_n; T_1, \ldots, T_n; u_1, \ldots, u_n\}$, where for each player $i \in I$, the non-empty set X_i is the strategy set, $T_i : X = \prod_{i\in I} X_i \to 2^{X_i}$ is the player's constraint correspondence, and $u_i : X \to R$ is the i-th player's payoff function. Whenever the player's constraint correspondence $T_i(x) = X_i$ for all $x \in X$ and all $i \in I$, the generalized game is reduced to $2n$-tuple $\Gamma = \{X_1, \ldots, X_n; u_1, \ldots, u_n\}$ and is called an N-person game of normal form. The set X is the Cartesian product of the individual strategy spaces. Denote by $X_{\hat{i}} = \prod_{j\in I\setminus\{i\}} X_j$. Denote by x_i and $x_{\hat{i}}$ an element of X_i and $X_{\hat{i}}$, respectively. Denote an arbitrary point of X by $x = (x_i, x_{\hat{i}})$, with x_i in X_i and $x_{\hat{i}}$ in $X_{\hat{i}}$. Let J be a non-empty subset of I. Denote $X_J = \prod_{i\in J} X_i$ and $X_{\hat{J}} = \prod_{i\in I\setminus J} X_i$. Denote by x_J and $x_{\hat{J}}$ an element of X_J and $X_{\hat{J}}$, respectively. Denote an arbitrary point of X by $x = (x_J, x_{\hat{J}})$, with x_J in X_J and $x_{\hat{J}}$ in $X_{\hat{J}}$. Denote $T : X \to 2^X$ by $T(x) = \prod_{i=1}^n T_i(x)$ for all $x \in X$.

A strategy vector $\tilde{x} \in X$ is said to be a *Nash equilibrium* for the generalized N-person game Γ if for each $i \in I$

$$\tilde{x}_i \in T_i(\tilde{x}) \quad \text{and} \quad u_i(\tilde{x}_i, \tilde{x}_{\hat{i}}) \geq u_i(x_i, \tilde{x}_{\hat{i}}) \quad \text{for all } x_i \in T_i(\tilde{x}).$$

A strategy vector $\tilde{x} \in X$ is said to be an *S-Nash equilibrium* for the generalized N-person game Γ if $\tilde{x}$ is a Nash equilibrium for Γ and

$$\sum_{i=1}^{n} u_i(x) \leq \sum_{i=1}^{n} u_i(\tilde{x}, x_{\hat{i}}) \quad \text{for all } x \in T(\tilde{x}).$$

Whenever the player's constraint correspondence $T_i(x) = X_i$ for all $x \in X$ and $i \in I$, a Nash equilibrium $\hat{x} \in X$ is said to be a *pure-strategy Nash equilibrium* for the N-person game Γ of normal form; an S-Nash equilibrium is said to be an *S-Nash-strategy equilibrium* for the N-person game Γ of normal form.

Let $J \subseteq I$. A strategy vector $\tilde{x}_J \in X_J$ is said to be a *J-dominant-strategy* if

$$u_i(\tilde{x}_i, x_{\hat{i}}) \geq u_i(x_i, x_{\hat{i}}) \quad \text{for all } x \in X \text{ and } i \in J.$$

A strategy vector $\tilde{x} \in X$ is said to be a *J-dominant-strategy Nash equilibrium* for the N-person game Γ of normal form; if $\tilde{x}$ is a pure-strategy Nash equilibrium and $\tilde{x}_J$ is a J-dominant-strategy.

3. Inequality

Let X be a topological space, and $A, Y \subseteq X$, A function $f : X \times Y \to R$ is $\mathcal{C}$-quasiconcave on A (see [11]) if, for any finite subset $\{x^0, \ldots x^n\}$ of A, there exists a continuous function $\phi_n : \triangle_n \to Y$ such that

$$f(\phi_n(\lambda), \phi_n(\lambda)) \geq \min_{i\in I(\lambda)} f(x^i, \phi_n(\lambda))$$

for all $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_n) \in \triangle_n$, where $I(\lambda) = \{i \mid \lambda_i \neq 0\}$.

We extend the above concept to a general $\mathcal{C}$-quasiconcave condition in the following:

Definition 3.1. Let X be a non-empty set and Y be a topological space, and $A \subseteq X$. A function $f : X \times Y \to R$ is said to be 0-*pair-concave on A*, if for arbitrary finite points $\{x^0, \ldots x^n\} \subset A$ are given, there is a continuous function $\phi_n : \triangle_n \to Y$ such that

$$\min_{i\in I(\lambda)} f(x^i, \phi_n(\lambda)) \leq 0$$

for all $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_n) \in \triangle_n$, where $I(\lambda) = \{i \mid \lambda_i \neq 0\}$.

Using Propositions 1 and 2 in [11] show that the $\mathcal{C}$-quasiconcavity unifies the diagonal transfer quasiconcavity (weaker than quasiconcavity) [5] and the $\mathcal{C}$-concavity (weaker than concavity) [7]. For the 0-pair-concavity, we have the following proposition that 0-pair-concavity unifies the $\mathcal{C}$-quasiconcavity [11].

Proposition 3.1. *Let X be a topological space, and $A, Y \subseteq X$. A function $f : X \times Y \to R$ is $\mathcal{C}$-quasiconcave on A. Define $U : X \times Y \to R$ by $U(x, y) = f(x, y) - f(y, y)$ for all $(x, y) \in X \times Y$. Then U is* 0*-pair-concave on A.*

Proof. Let $\{x^0, \ldots, x^n\}$ be a finite subset of A. Since f is $\mathcal{C}$-quasiconcave on A, there is a continuous function $\phi_n : \triangle_n \to Y$ such that

$$f(\phi_n(\lambda), \phi_n(\lambda)) \geq \min_{i\in I(\lambda)} f(x^i, \phi_n(\lambda))$$

for all $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_n) \in \triangle_n$, where $I(\lambda) = \{i \mid \lambda_i \neq 0\}$. Then

$$\min_{i\in I(\lambda)} U(\hat{x}^i, \phi_n(\lambda)) \leq 0$$

for all $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_n) \in \triangle_n$, where $I(\lambda) = \{i \mid \lambda_i \neq 0\}$. This completes the proof. $\square$