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On the existence of Nash equilibriums for infinite matrix games

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Abstract

The fundamental theorem of game theory states that for every matrix game with finite strategies there exists at least one optimal strategy. It is known that the fundamental theorem of game theory does not hold, in general, in infinite matrix games. In this paper, we provide a characteristic of Nash equilibriums for $\infty \times \infty$ matrix games and prove an existence theorem of optimal strategies by using the Fan–KKM theorem. We give some applications of these theorems.

Keywords: $\infty \times \infty$ matrix games; Nash equilibrium; Optimal strategy; Saddle point; KKM mapping; Fan–KKM theorem

1. Introduction

Let $A = (a_{ij})_{m,n}$ be the pay-off matrix of an $m \times n$ matrix game, where m and n are positive integers, in which the two players are named player R (row player) and player C (column player). Suppose that player R has strategies $r_1, r_2, \ldots, r_m$ and player C has strategies $c_1, c_2, \ldots, c_n$. For any $1 \leq i \leq m$, $1 \leq j \leq n$, a_{ij} is the pay-off for player R when player R applies strategy r_i and player C applies strategy c_j. Let $X = (x_1, x_2, \ldots, x_m)$ be a probability distribution for player R to apply his strategies $r_1, r_2, \ldots, r_m$, and $Y = (y_1, y_2, \ldots, y_n)^{\mathrm{T}}$ be a probability distribution for player C to apply his strategies $c_1, c_2, \ldots, c_m$. Let $E(X, Y) = XAY$, which is the expected value of the game with respect to the probability distributions X and Y. From the well-known Fundamental Theorem of Game Theory (see [1,6]), there exists a pair of probability distributions X^* and Y^* for players R and C, respectively, such that

$$E(X, Y^*) \leq E(X^*, Y^*) \leq E(X^*, Y), \tag{0}$$

for every probability distribution X and Y for players R and C, respectively. Then (X^*, Y^*) is called a Nash equilibrium, or an optimal strategy, or a saddle point of this game and $E(X^*, Y^*)$ is called the value of this game. The game is said to be fair if its value $E(X^*, Y^*) = 0$.

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For every given $m \times n$ matrix game A, the expected value function $E(X, Y)$ is a function of X and Y defined on $S^{m-1} \times S^{n-1}$, where $S^{k-1} = \{(x_1, x_2, \ldots, x_k) \in R^k : x_1, x_2, \ldots, x_k \geq 0 \text{ and } x_1 + x_2 + \cdots + x_k = 1\}$ is the k-dimensional unit simplex, for $k = m$ or n. We know that $S^{m-1} \times S^{n-1}$ is a convex and compact subset of $R^m \times R^n$. Based on the compactness of $S^{m-1} \times S^{n-1}$, the Fundamental Theorem of Game Theory can be induced by the well-known Von Neumann Theorem that can be proved by the known Kakutani Fixed Point Theorem (see [6]). In infinite-dimensional cases, let $S = \{(x_1, x_2, \ldots) : x_i \geq 0, i = 1, 2, \ldots \text{ and } \sum_{i=1}^{\infty} x_i = 1\}$. $S \times S$ is a convex but not a compact subset of $l^p \times l^p$, for any $p \geq 1$. And therefore the Von Neumann Theorem cannot be analogously extended to infinite-dimensional cases. Hence, the Fundamental Theorem of Game Theory does not hold in infinite-dimensional cases. So the conditions to assure the existence of equilibrium for a given infinite matrix game turns out to be an interesting problem in game theory. In [2–6], Radzik, Mendez-Naya, Wald and others investigated the existence properties and gave some conditions for the existence of equilibrium for infinite matrix games.

In this paper, we study the existence properties for $\infty \times \infty$ matrix games. In Section 2, we list a characteristic of equilibriums for $\infty \times \infty$ matrix games which can be used to test if a $\infty \times \infty$ matrix game has Nash equilibrium or not. In Section 3, we prove an existence theorem of optimal strategies for $\infty \times \infty$ matrix games by using the Fan–KKM theorem. In Section 4, we give some examples as applications of these theorems.

2. A characteristic of equilibriums for $\infty \times \infty$ matrix games

In an $\infty \times \infty$ matrix games, suppose that player R has infinite strategies $r_1, r_2, \ldots$ and player C has infinite strategies $c_1, c_2, \ldots$. Let $A = (a_{ij})$ be the pay-off matrix of this matrix game. It is an $\infty \times \infty$ matrix. A pair of strategies (r_k, c_l) is said to be a saddle strategy of this matrix game if a_{kl} is the minimum entry in the kth row and the maximum entry in the lth column of matrix A. A game is said to be strictly determined if it has at least one saddle strategy.

Similar to the finite-dimensional case, if an $\infty \times \infty$ matrix game is not strictly determined, then the players want to find probability distributions (mixed strategies) to apply their strategies so that player R can maximize his wins and player C can minimize his loses if this game will be independently repeated to play. Let $S = \{(x_1, x_2, \ldots) : x_i \geq 0, i = 1, 2, \ldots \text{ and } \sum_{i=1}^{\infty} x_i = 1\}$. It is the infinite-dimensional simplex. It is known that S is a closed convex subset of l^2. But it is not compact. For every $\infty \times \infty$ matrix $A = (a_{ij})$, let $\|A\| = \sup\{|a_{ij}| : 1 \leq i, j < \infty\}$. We define $E(X, Y) = XAY^{\mathrm{T}}$, for all $(X, Y) \in S \times S$, where without confusing A is omitted in $E(X, Y)$. It is known that $E(\cdot, \cdot)$ is well-defined on $S \times S$ if and only if $\|A\| < \infty$.

We give a characteristic of equilibriums theorem below. Similar result may be found in some related research. For the application of the following contents and for the readers convenience, we give the proof of the following theorem.

Theorem 1. *Let $A = (a_{ij})$ be the $\infty \times \infty$ pay-off matrix of a given infinite matrix game satisfying $\|A\| < \infty$. Let $X^* = (x_i^*)$ and $Y^* = (y_i^*)$ be a pair of probability distributions. Then $(X^*, Y^*) \in S \times S$ is a Nash equilibrium of this game if and only if the following equality holds*

$$\sup_m \sum_{j=1}^{\infty} a_{mj} y_j^* = \inf_n \sum_{i=1}^{\infty} a_{in} x_i^*. \tag{1}$$

Proof. ($\Rightarrow$) Suppose $(X^*, Y^*) \in S \times S$ is a Nash equilibrium of the function $E(\cdot, \cdot)$. Then for any $(X, Y) \in S \times S$, we have

$$E(X, Y^*) \leq E(X^*, Y^*) \leq E(X^*, Y),$$

that is

$$\sum_{i,j=1}^{\infty} a_{ij} x_i y_j^* \leq \sum_{i,j=1}^{\infty} a_{ij} x_i^* y_j^* \leq \sum_{i,j=1}^{\infty} a_{ij} x_i^* y_j.$$

For any positive integers m and n, taking $X, Y \in S$ such that the mth entry of X and the nth entry of Y are 1 and all other entries are 0, we have

$$\sum_{j=1}^{\infty} a_{mj} y_j^* \leq \sum_{i,j=1}^{\infty} a_{ij} x_i^* y_j^* \leq \sum_{i=1}^{\infty} a_{in} x_i^*, \quad \text{for all } m, n = 1, 2, 3, \ldots.$$