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Interim Bayesian Nash equilibrium on universal type spaces for supermodular games ☆

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Abstract

We prove the existence of a greatest and a least interim Bayesian Nash equilibrium for supermodular games of incomplete information. There are two main differences from the earlier proofs and from general existence results for non-supermodular Bayesian games: (a) we use the interim formulation of a Bayesian game, in which each player's beliefs are part of his or her type rather than being derived from a prior; (b) we use the interim formulation of a Bayesian Nash equilibrium, in which each player and every type (rather than almost every type) chooses a best response to the strategy profile of the other players. There are no restrictions on type spaces and action sets may be any compact metric lattices.

JEL classification: C72; C61

Keywords: Supermodular games; Incomplete information; Universal type spaces; Interim Bayesian Nash equilibrium

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1. Introduction

We prove the existence of a greatest and a least *interim* pure-strategy Bayesian Nash equilibrium for supermodular games of incomplete information. Here is a summary of the main result. Consider the following:

- the *interim* formulation of a Bayesian game, with type spaces and with each individual's beliefs given by a mapping from her set of types to beliefs about the other players' types and about the state of nature (in contrast to the ex ante formulation, in which beliefs are given by a common prior and conditional beliefs);
- the *interim* formulation of a Bayesian Nash equilibrium, in which each player and each type maximizes her expected payoff (in contrast to the ex ante formulation as the Nash equilibrium of an ex ante game, in which each player maximizes her expected payoff for almost every type).

We place no assumptions on each set of types, other than that it is endowed with a sigma-field. Suppose that the following hold for each player:

(1) her set of actions is a compact metric lattice;
(2) her payoff is measurable in the types, continuous in the actions, bounded, and supermodular in own action, and has increasing differences between her own action and the other players' actions;
(3) her interim belief about each event is measurable in her own type.

Then there exist a greatest and a least pure-strategy equilibria.

Vives [20] and Milgrom and Roberts [15] also obtain existence of a pure-strategy Bayesian Nash equilibrium for supermodular games of incomplete information. The main differences are that (a) we use the interim formulation of a Bayesian game, in which each player's beliefs are part of his or her type rather than being derived from a prior; (b) we use the interim formulation of a Bayesian Nash equilibrium, in which each player and every type (rather than almost every type) chooses a best response to the strategy profile of the other players; (c) we assume that the action spaces are compact metric lattices, whereas they assume that each action set is a compact sublattice of Euclidean space. The proof in [15] applies a general existence theorem for supermodular games to the ex ante normal form of the Bayesian game; the proof in [20] uses a Cournot tatônnement that is also the basis of the proof in this paper.

Like those authors, we are exploiting the fact that the game is supermodular. Because there is uncertainty and players maximize expected payoffs, each player's underlying payoff function must be truly supermodular in own action and have increasing differences between own action and other actions (properties that are preserved by integration) rather than merely satisfy ordinal quasi-supermodularity and single-crossing properties (which are not preserved by integration). For example, a Bayesian game with log-supermodular payoffs need not have a pure-strategy equilibrium.

There are various papers on existence of BNE in *monotone-in-type* pure strategies, in which the type spaces are also partially ordered (unlike in this paper). Since monotone-in-type is a stronger property than we seek here, those papers need additional assumptions on complementarity between actions and types and on monotonicity of beliefs. Leaving these assumptions aside, here is how the models compare. Van Zandt and Vives [19] use the same set-up as in this