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A general structure theorem for the Nash equilibrium correspondence

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ABSTRACT

I consider n-person normal form games where the strategy set of each player is a non-empty compact convex subset of an Euclidean space, and the payoff function of player i is continuous in joint strategies and continuously differentiable and concave in the player i's strategy. No further restrictions (such as multilinearity of the payoff functions or the requirement that the strategy sets be polyhedral) are imposed. I demonstrate that the graph of the Nash equilibrium correspondence on this domain is homeomorphic to the space of games. This result generalizes a well-known structure theorem in [Kohlberg, E., Mertens, J.-F., 1986. On the strategic stability of equilibria. Econometrica 54, 1003–1037]. It is supplemented by an extension analogous to the unknottedness theorems in [Demichelis S., Germano, F., 2000. Some consequences of the unknottedness of the Walras correspondence. J. Math. Econ. 34, 537–545; Demichelis S., Germano, F., 2002. On (un)knots and dynamics in games. Games Econ. Behav. 41, 46–60]: the graph of the Nash equilibrium correspondence is ambient isotopic to a trivial copy of the space of games.

1. Introduction

This paper contributes to the study of the geometry of Nash equilibria. The existing studies of the geometry of Nash equilibria usually consider mixed strategy Nash equilibria on the domain of finite games, i.e. games with finitely many pure strategies. This framework, however, is inadequate for modelling a large number of interesting strategic interactions. Market games, Cournot oligopoly games, location games are examples of games with a continuum of pure strategies and non-linear payoffs. In such games, it is the set of equilibria in pure rather than mixed strategies that is of particular importance to the respective applications. The purpose of this paper is to extend the study of the geometry of Nash equilibria on a sufficiently rich domain of games that includes these and similar types of games.

The geometry of Nash equilibria is best understood through the properties of the graph of the equilibrium correspondence. A number of topological characterizations of the graphs of various equilibrium correspondences are well known in the literature. Thus Kohlberg and Mertens (1986) show that the graph of the Nash equilibrium correspondence on the domain of finite games is homeomorphic to an Euclidean space. This result is a game-theoretic analogue of the structure theorem in Balasko (1978) who shows that the graph of the Walrasian equilibrium correspondence is homeomorphic to an Euclidean space. A topological characterization of the pseudo-equilibrium manifold in economies with incomplete markets is given in Zhou (1997). In Demichelis et al. (2004) the graph of subgame perfect equilibrium correspondence is shown to be homeomorphic with the underlying space of perfect information games.

This paper provides a topological characterization of the Nash equilibrium correspondence in a very general setup. I consider the space of normal form games as parameterized by the payoff functions. It is assumed that the strategy set of each

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player is a non-empty compact convex subset of an Euclidean space, and the payoff function of player i is continuous in joint strategies and concave and continuously differentiable in the own strategies of player i. No further restrictions (such as multilinearity of the payoff functions or the requirement that the strategy sets be polyhedral) are imposed. It is demonstrated that the graph of the Nash equilibrium correspondence on this domain is homeomorphic to the underlying space of games. Furthermore, the constructed homeomorphism preserves the subspace of finite games, implying (the first part of) the structure theorem in Kohlberg and Mertens (1986) as a corollary.

Multiplicity of equilibria is the main reason why the topological characterizations such as the one in the present paper are not easy to obtain. Even in one-player games as considered in Section 2.1, where Nash equilibrium is simply a maximum of a payoff function, the characterization is non-trivial due to the possibility of multiple maxima. And in games with two or more players there may be many Nash equilibria even if the best responses are always unique. The second source of multiplicity of equilibria is the strategic interrelatedness of the players' best responses.

It turns out that the geometry of Nash equilibrium in decision problems (i.e. one-player games) is similar to the geometry of subgame perfect equilibrium on the domain of perfect information games (see Demichelis et al., 2004). Indeed, the only source of multiplicity of equilibria in such games is the indeterminateness of the best response due to indifferences in the preferences over terminal outcomes. In both cases the equilibrium correspondence is almost everywhere a continuous single-valued function that occasionally makes a vertical step and even a small perturbation of the ambient space is sufficient to make the graphs of these correspondences look like the graphs of single-valued functions. These results cannot be extended to games with many players. In the general case, only a sufficiently large perturbation of the ambient space can deform the graph of the Nash equilibrium correspondence to a graph of a single-valued map.

In Section 3 I develop an extension of the structure theorem in the spirit of the so-called unknottedness theorem in Demichelis and Germano (2000, 2002). I show that not only does the graph of the Nash equilibrium correspondence have an intrinsic structure of the space of games, but it can be continuously deformed within its ambient space (games times strategies) to a graph of a single-valued function. It follows as a corollary that the homeomorphism of the graph of the Nash equilibrium correspondence with games is proper homotopic with the projection map, a result analogous to the second part of the structure theorem in Kohlberg and Mertens (1986).

For finite games the unknottedness result is known to have a number of important implications for the dynamics whose rest points are equilibria. For example, it implies that any two Nash dynamics are homotopic within the set of Nash dynamics and that the degree and the index of any two equilibria are equal. Extending these results to larger domains of games such as that studied in this paper is an interesting direction for future research.

2. The structure theorem

It is well known that games have many Nash equilibria. It is in fact the multiplicity of equilibria that makes the structure theorem a non-trivial result. One can distinguish two sources of multiplicity of equilibria, the first source being the multiplicity of the best responses. This source of multiplicity is present even in decision problems (i.e. one-player games). And in games with two or more players there may be many Nash equilibria even if the best responses are always unique. The second source of multiplicity of equilibria is the strategic interrelatedness of the players' best responses. We shall be dealing with each source of multiplicity of equilibria in turn, first considering one-player games and then turning to the general case.

2.1. The one-player case

There is a set X of strategies that is assumed to be a non-empty, concave and convex subset of a finite-dimensional Euclidean space. The set X will be fixed, and the one-player games will be parameterized by the payoff functions $u : X \to \mathbb{R}$. Each payoff function is assumed to be continuous and concave. Let $\mathscr{U}$ be the collection of all such functions. A Nash equilibrium of a (one-player) game u is a strategy that maximizes the payoff function u on the set X. The Nash equilibrium correspondence assigns to each payoff function u in $\mathscr{U}$ the set of Nash equilibria of the game u. Thus the graph of the Nash equilibrium correspondence is the set

$$\mathscr{N} = \big\{(u, x) \in \mathscr{U} \times X \mid x \text{ maximizes } u \text{ on } X\big\}.$$

For a given function $u \in \mathscr{U}$ the set of maximizers of u is a convex set, and it is a singleton if u is strictly convex. Thus the Nash equilibrium correspondence is almost always a continuous single-valued function that occasionally makes a vertical step. A vertical step occurs whenever a payoff function has a flat section that produces multiple maxima. Panel (a) of Fig. 1 depicts the Nash equilibrium correspondence on the domain of all linear payoff functions when X is an interval $[-1, 1]$.

Let $\eta : \mathscr{N} \to \mathscr{U}$ be defined by the equation $\eta(u, x) = u + l_x$, where l_x is a linear function given by the formulae $l_x(z) = \langle x, z\rangle$. Here $\langle x, z\rangle$ denotes the inner product of the vectors x and z. Notice that the function η preserves the linearity of the payoff function: if u is a linear function, so is $\eta(u, x)$.

To illustrate how the map η works, consider the case where $X = [-1, 1]$ and consider the collection of all linear functions $\{l_a\}$ as parameterized by the slope a. The graph of the Nash equilibrium correspondence on the domain of all linear functions is depicted in panel (a) of Fig. 1. The Nash equilibrium correspondence is single-valued at all points except $a = 0$ where it