Available online at www.sciencedirect.com
ScienceDirect
ELSEVIER
Nonlinear Analysis 66 (2007) 1645–1659
Nonlinear Analysis
www.elsevier.com/locate/na

On extremal pure Nash equilibria for mixed extensions of normal-form games

S. Heikkilä*

Department of Mathematical Sciences, University of Oulu, Box 3000, FIN-90014 University of Oulu, Finland

Received 12 December 2005; accepted 17 February 2006

Abstract

In this paper we derive conditions under which mixed extensions of normal-form games have least and greatest Nash equilibria in pure strategies, and either of them gives best utilities among all mixed Nash equilibria when strategy spaces are complete separable metric spaces equipped with closed partial orderings, and the values of utility functions are in separable ordered Banach spaces. The obtained results are applied to supermodular normal-form games whose strategy spaces are multidimensional.

Keywords: Game; Normal-form; Supermodular; Strategy; Pure; Mixed; Nash equilibrium; Greatest; Least; Ordered Polish space; Ordered Banach space; Generalized iteration method

1. Introduction

In this paper existence and comparison results are derived for extremal Nash equilibria to the mixed extension Γ of a normal-form game $G = \{S_1, \ldots, S_N, u_1, \ldots, u_N\}$ when the strategy spaces S_i are complete separable metric spaces equipped with closed partial orderings, and the values of utility functions u_i are in ordered separable Banach spaces.

Difficulties which are encountered when the strategy spaces of a supermodular normal-form game are not in $\mathbb{R}$ are described in Section 3 of [5] as follows: "When the strategy spaces are multidimensional, the set of mixed strategies is not a lattice. This implies that we lack the mathematical structure needed for the theory of complementarities. Multiple best responses are always present when dealing with mixed equilibria and there does not seem a simple solution to the requirement that strategy spaces be lattices".

* Tel.: +358 85531736; fax: +358 85531730.
E-mail address: sheikki@cc.oulu.fi.

0362-546X/$ - see front matter
doi:10.1016/j.na.2006.02.015

As shown in Section 3, the lack of lattice structure for mixed strategies does not prevent us from proving that the mixed extension Γ of a normal-form game G has least and greatest Nash equilibria in pure strategies, and that under suitable monotonicity hypotheses either of their utilities majorizes the utilities of all Nash equilibria for Γ. These results are shown in Section 4 to hold true also for mixed extensions of supermodular games whose strategy spaces are multidimensional, which extends the corresponding results derived in [5] when the strategy spaces are in $\mathbb{R}$. The arguments given in Remark 1 justify that the generalized iteration methods introduced in [6] play an indispensable role in the proofs.

2. Preliminaries

By a *partially ordered set* (*poset*) we mean a nonempty set X equipped with a reflexive, antisymmetric and transitive order relation "$\leq$". If $x \leq y$ and $x \neq y$, denote $x < y$. A sequence (x_n) of X is called *increasing* if $x_n \leq x_m$ whenever $n \leq m$, *decreasing* if $x_m \leq x_n$ whenever $n \leq m$, and *monotone* if (x_n) is increasing or decreasing. A function f from a poset $X = (X, \leq)$ to another poset $Y = (Y \preceq)$ is called *increasing* if $f(x) \preceq f(y)$ whenever $x \leq y$, *decreasing* if $f(y) \preceq f(x)$ whenever $x \leq y$, and *monotone* if f is increasing or decreasing.

Let S be a nonempty subset of a poset $X = (X, \leq)$. If $z \in X$ and $x \leq z$ for all $x \in S$, then z is called an *upper bound* of S. If $z \leq y$ for all other upper bounds y of S, we say that z is the *least upper bound* of S, and denote $z = \sup S$. If $z = \sup S \in S$, we say that z is the *greatest element* of S, and denote $z = \max S$. A lower bound, a greatest lower bound $\inf S$ and a least element $\min S$ of S are defined similarly.

We say that a nonempty subset S of a poset X is *directed upward* if to each pair x, y of elements of S there corresponds a $z \in S$ such that $x \leq z$ and $y \leq z$. If the reversed inequalities hold, we say that S is *directed downward*. If S is both upward and downward directed, we call S *directed*. S is called a *join sublattice* of X if $x \vee y := \sup\{x, y\}$ exist in X and belongs to S for all $x, y \in S$. If $x \wedge y := \inf\{x, y\}$ exist in X and belongs to S for all $x, y \in S$, we say that S is a *meet sublattice*. If S both meet and join sublattice, we say that S is a *sublattice*. S is a *chain* if $x \leq y$ or $y \leq x$ for all $x, y \in S$. Every chain is a sublattice and every sublattice is directed.

Let X be a real vector space, equipped with a partial ordering $\leq$. If $x \leq y$ implies that $x + z \leq y + z$ for all $z \in X$ and $\alpha x \leq \alpha y$ for all $\alpha \geq 0$, we say that X is an *ordered vector space*. The set $X^+ = \{x \in X \mid 0 \leq x\}$ is called an *order cone* of X. If X is equipped with a Banach norm $\|\cdot\|$, and if X^+ is a closed subset of $X = (X, \|\cdot\|)$, we say that $X = (X, \|\cdot\|, \leq)$ is an *ordered Banach space*. The order cone X^+ is called *regular* if all order bounded and increasing sequences of X^+ converge. As for examples of ordered Banach spaces with regular order cones see, e.g., [6, Subsections 1.3 and 5.8].

By an *ordered metric space* we mean a metric space $S = (S, d)$ equipped with such a partial ordering $\leq$ that the order intervals $[a) = \{x \in S \mid a \leq x\}$ and $(b] = \{x \in S \mid x \leq b\}$, and hence also $[a, b] = [a) \cap (b]$, are closed subsets of S for all $a, b \in S$. We say that a nonempty subset A of an ordered metric space is *closed upward* if it contains the limits of all its increasing and convergent sequences. If A contains the limits of all its decreasing and convergent sequences, we say that A is *closed downward*. A is called *order closed* if A is closed upward and downward.

We say that an ordered metric space $S = (S, d, \leq)$ is an *ordered Polish space* if S is complete and separable, and if the partial ordering $\leq$ is *closed* in the sense that if $x_n \to x$, $y_n \to y$ and $x_n \leq y_n$ for each n, then $x \leq y$. For instance, nonempty and closed subsets of separable ordered Banach spaces are ordered Polish spaces.