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Convergence to approximate Nash equilibria in congestion games ☆

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ABSTRACT

We study the ability of decentralized, local dynamics in non-cooperative games to rapidly reach an approximate (pure) Nash equilibrium. Our main result states that for symmetric congestion games in which the cost function associated with each resource satisfies a "bounded jump" condition, convergence to an ε-Nash equilibrium occurs within a number of steps that is polynomial in the number of players and ε^{-1}. We show moreover that this result holds under a variety of conventions governing the move orders among the players, including the minimal liveness assumption that no player is indefinitely blocked from moving. We also prove that in the generalized setting where players have different "tolerances" ε_i, the number of moves a player makes before equilibrium is reached depends only on his associated ε_i, and not on those of other players. Finally, we show that polynomial time convergence holds even when a constant number of resources have arbitrary cost functions.

1. Introduction

The emerging field of algorithmic game theory has led to a fundamental re-examination, from a computational perspective, of the classical concept of Nash equilibrium (Nash, 1950). Much of this activity has focused on understanding the properties of Nash equilibria (as expressed, notably, in the "price of anarchy," see e.g. Papadimitriou, 2001; Roughgarden and Tardos, 2004; Roughgarden, 2005) and the computational complexity of finding them (see, e.g., Fabrikant et al., 2004; Daskalakis et al., 2006; Chen and Deng, 2006). Considerably less is understood about the question of whether selfish players, acting in a decentralized fashion, actually arrive at a Nash equilibrium in a reasonable amount of time. This would seem to be a central consideration in the computational study of Nash equilibria.

In this paper we address this question in the general arena of *congestion games*. A congestion game is an n-player game in which each player's strategy consists of a set of resources, and the cost of the strategy depends only on the number of players using each resource, i.e., the cost takes the form $\sum_e d_e(f(e))$, where $f(e)$ is the number of players using resource e, and d_e is a non-negative increasing function. A standard example is a *network* congestion game on a directed graph, in which each player must select a path from some source to some destination, and each edge has an associated "delay" function that increases with the number of players using the edge. In what follows, we shall use the terminology of edges and delays even though we will always be discussing general (non-network) congestion games.

Congestion games have attracted a good deal of attention, partly because they capture a large class of routing and resource allocation scenarios, and not least because they are known to possess *pure* Nash equilibria (Rosenthal, 1973). Thus

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unlike general games, whose Nash equilibria may involve mixed (i.e., randomized) strategies for the players, congestion games always have a Nash equilibrium in which each player sticks to a single strategy. Further, in congestion games, the natural decentralized mechanism known as the "Nash dynamics," in which at each step some player switches her strategy to a better alternative, is guaranteed to converge to a pure Nash equilibrium. The question then is the following: *Starting from an arbitrary initial state, does the Nash dynamics converge rapidly*?

The work of Fabrikant et al. (2004) provides a devastating negative answer, even for symmetric[2] congestion games: the problem of finding a Nash equilibrium is *PLS-complete* (Johnson et al., 1988), and therefore as difficult as that of finding a local optimum in any local search problem with efficiently computable neighborhoods. Moreover, there are examples of games and initial strategies such that the shortest path to an equilibrium in the Nash dynamics is exponentially long in the number of players n. Thus if we want a notion of Nash equilibrium that is selfishly and efficiently realizable, the best we can hope for is some kind of approximation. (Indeed, given the recent spate of hardness results for finding exact equilibria in most classes of games by *any* algorithmic means (Fabrikant et al., 2004; Daskalakis et al., 2006; Chen and Deng, 2006), it seems inevitable that attention will now shift to approximation.)

Accordingly, we say that a state s (i.e., a collection of strategies for the players) is an *ε-Nash equilibrium* if no player can improve her cost by more than a factor of ε by unilaterally changing her strategy. This definition has intuitive appeal, for example, if one imagines charging players a percentage of their current cost for the privilege of changing strategy.[3] Given this definition, we introduce a natural modification of the Nash dynamics called the *ε-Nash dynamics*, which permits only *ε-moves*, i.e., moves that improve the cost of the player by a factor of more than ε. Clearly ε-Nash equilibria correspond to fixed points of this dynamics. Our goal is to investigate under what circumstances the ε-Nash dynamics does in fact converge rapidly to an ε-Nash equilibrium.

To make the ε-Nash dynamics concrete, we assume that among multiple players with ε-moves available, at each step a move is made by the player with the largest incentive to move—i.e., the player who can make the largest relative improvement in cost—(with ties broken arbitrarily), and that this player actually makes this best move. This is a minimal coordination mechanism that seems natural in our context; however, as we shall see later, our results hold even with *no* coordination under only a basic liveness assumption.

In order to state our results we need one further notion. For any $\alpha \geqslant 1$, we say that an edge in a congestion game satisfies the *α-bounded jump* condition if its delay function satisfies $d_e(t+1) \leqslant \alpha d_e(t)$ for all $t \geqslant 1$. We will think of α as being a constant, or at most polynomially bounded in n. The bounded jump condition means that when a new player is added to an edge, the cost to all players using that edge increases by at most a factor of α. This condition is rather weak (see below); in particular, an edge with $d_e(t) = \alpha^t$ satisfies the α-bounded jump condition.

We are now ready to state our first main result, which says that in any symmetric congestion game with bounded jumps, the ε-Nash dynamics converges rapidly to an ε-Nash equilibrium. This is apparently the first such result for such a broad class of (atomic) congestion games, and in particular for a class that contains PLS-complete examples.

Theorem 1.1. *In any symmetric congestion game with n players in which all edges satisfy the α-bounded jump condition, the ε-Nash dynamics converges from any initial state in $\lceil n\alpha\varepsilon^{-1}\log(nC)\rceil$ steps, where C is an upper bound on the cost of any player.*

The proof of this theorem relies on two key properties. First, the existence of an "exact" potential function (Rosenthal, 1973), whose decrease under any move reflects exactly the improvement in cost of the moving player. And second, the fact that, under the bounded jump condition, any player can emulate the move of any other with at most an α-factor overhead. This ensures that every move of the dynamics decreases the potential function by an $\frac{\varepsilon}{\alpha n}$ factor.

We now briefly discuss the bounded jump condition. First, as we show later (Section 3.1), the hardness results mentioned above for finding exact equilibria carry over to symmetric games in which all edges have α-bounded jumps. Second, the bounded jump condition is a rather weak assumption that is similar to conditions imposed in other quantitative studies of transient behavior (e.g., the "bounded relative slope" of Fischer et al., 2006 or "bounded slope" of Blum et al., 2006); it is much weaker than the polynomial bounds typically used in studies of the price of anarchy (Awerbuch et al., 2005; Christodoulou and Koutsoupias, 2005). Third, it is questionable how much sense it makes to talk about "symmetric" congestion games without such a condition. This is because of the trick in Fabrikant et al. (2004) (see Section 3.1 below) for making any congestion game symmetric by adjoining to the strategies of each player p_i a special edge e_i whose delay is small for one player and huge for more than one player. This effectively divides the strategies into sets, one player per set, and is equivalent to the original game up to a relabeling of the players. Thus, if we could prove Theorem 1.1 without the bounded jump condition, we would get rapid convergence for *all* congestion games.[4] The bounded jump condition can

[2] A *symmetric* game is one in which the allowed strategies of all the players are the same.

[3] An alternative notion of approximate equilibrium (see, e.g., Even-Dar et al., 2003; Fabrikant et al., 2004; Kearns and Mansour, 2002; Lipton et al., 2003) is based on an *additive* error of ε, rather than the relative error we use here. We would argue that our definition is equally natural, and indeed more in line with approximation guarantees in Computer Science and also with the definition of price of anarchy (Papadimitriou, 2001).

[4] Recently, following the appearance of a preliminary version of this paper (Chien and Sinclair, 2007), Skopalik and Vöcking (2008) showed that this is not possible by demonstrating the existence of asymmetric congestion games (with the bounded jump condition) for which the ε-Nash dynamics necessarily requires an exponential number of steps to converge from certain initial states.