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Coalition-proof Nash equilibria and cores in a strategic pure exchange game of bads

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Abstract

We present an example of a strategic game in which the major solutions allowing coalitions always exist. Specifically, we consider the pure exchange game due to Scarf [Scarf, H.E., 1971. On the existence of a cooperative solution for a general class of n-person games. Journal of Economic Theory 3, 169–181] with all the commodities being replaced by bads. It is shown that the coalition-proof Nash equilibrium describes the behavior with every player dumping all the initial bads onto just one next player unilaterally. On the other hand, the α-core also exists and coincides with the β-core without any convexity assumptions. If, in particular, the bads are of one type, any coalition-proof Nash equilibrium reduces to a strong Nash equilibrium; and every player retaining all the initial bads is in the α-core if and only if the distribution of the initial bads is moderate without too 'big' players.

Keywords: Coalition-proof Nash equilibria; Strong Nash equilibria; α-Core; β-Core

JEL classification: C72; C71

1. Introduction

As is well-known, the strong Nash equilibrium in strategic games is a solution concept defined to be a strategy profile at which no coalition has a deviation. Also, the coalition-proof Nash equilibrium due to Bernheim et al. (1987) is a refined concept of the Nash equilibrium without *credible* deviations; namely, without deviations that can be regarded not

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to trigger further subcoalitional deviations. This of course implies that the strong Nash equilibrium is a special case of the coalition-proof Nash equilibrium. On the other hand, the α-core and its subset, the β-core, are also well-known solution concepts in strategic games with coalitions.

These solutions involving coalitions are, however, generally hard to obtain except for the α-core of a game with certain convexity assumptions. In the literature, several attempts have been made to obtain coalition-proof Nash equilibria. For example, allowing correlated strategies, Moreno and Wooders (1996) found a sufficient condition via the iterated elimination of dominated strategies. Milgrom and Roberts (1996) considered a game with strategic complementarity and formulated a sufficient condition in terms of monotone externalities. Konishi et al. (1997), Kukushkin (1997) and Yi (1999) presented existence results in games each with a specific structure.

We do not, however, deal with the general existence problem in this paper; instead, we shall present an example of a strategic game that always has coalition-proof Nash equilibria without any special assumptions and, moreover, has the nonempty α-core that coincides with the β-core without any convexity assumptions. Specifically, we consider the pure exchange game due to Scarf (1971) with all the commodities being replaced by bads. This game is not contained in the classes considered by the authors cited above; but, may be viewed as a general, strategic version of the garbage disposal TU game discussed by Shapley and Shubik (1969), so that we might call the game the strategic garbage disposal game.

It will be shown that at any Nash equilibrium, every player dumps all the initial holding of bads onto the other players. Interestingly, given any permutation σ of all players $i=1, \ldots, n$, the strategy profile such that, for each i with $n+1\equiv 1$, player $\sigma(i)$ dumps all the bads onto the player $\sigma(i+1)$ plays a key role. In fact, such a strategy profile is shown to be coalition-proof unless it is weakly dominated by a strategy profile of the same form under a different permutation. No other strategy profile is coalition-proof. On the other hand, the α-core is directly shown to be nonempty using the above strategy profile under σ and, moreover, is shown to coincide with the β-core. We will also show that the α-NTU game derived from the strategic garbage game is balanced without any convexity assumption.

If, in particular, the bads can be treated as a single garbage, any coalition-proof Nash equilibrium reduces to a strong Nash equilibrium. This is a remarkable result, since strong Nash equilibria are generally hard to obtain. The α-core, on the other hand, can describe just opposite behavior, namely, every player retaining the initial amount of the single garbage is in the α-core if and only if the distribution of the initial holding of garbage is moderate without too 'big' players. This result is in a sharp contrast to that of Shapley and Shubik (1969) with an empty TU core.

In the next section, we state definitions of solutions considered in this paper. Then, the garbage disposal game is investigated with results mentioned above.

2. The strategic game

The game in strategic form is a tuple $G=(N, \{X^i\}_{i\in N}, \{v^i\}_{i\in N})$, where $N=\{1, \ldots, n\}$ is the set of players, X^i is the set of strategies of $i\in N$, and v^i is the pay-off function of $i\in N$. Any nonempty subset $S\subseteq N$ will be called a coalition, and the singleton $\{i\}$ will be sometimes identified with i.