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The intrinsic comparative dynamics of locally differentiable feedback Nash equilibria of autonomous and exponentially discounted infinite horizon differential games ☆

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ABSTRACT

The comparative dynamics of locally differentiable feedback Nash equilibria are derived for the ubiquitous class of autonomous and exponentially discounted infinite horizon differential games. The resulting refutable implications are intrinsic to the said class of differential games, and thus form their basic, empirically testable, properties. Their relationship with extant results in the optimal control theory and the static game theory is discussed. Separability conditions are identified on the instantaneous payoff and transition functions under which the intrinsic comparative dynamics collapse, in form, to those in optimal control problems. Applications of the results to capital accumulation and sticky-price games are provided.

1. Introduction

The derivation of the basic, fundamental, or intrinsic comparative statics of an economic model—the archetypal utility maximization problem—can be traced back to Antonelli (1886) and Slutsky (1915). But it was Samuelson (1947, p. 32) who first derived the intrinsic comparative statics of the class of differentiable, unconstrained optimization problems. Twenty-seven years later Silberberg (1974) provided an extension of Samuelson's (1947) theorem by deriving the basic comparative statics of the class of differentiable, constrained optimization problems. More than three decades then passed before Partovi and Caputo (2006, 2007) further generalized and unified the approach to deriving the fundamental comparative statics of the class of differentiable, constrained optimization problems using the concept of generalized compensated derivatives.

It is important to remark at this juncture that the intrinsic comparative statics of differentiable optimization problems are defined as those qualitative comparative statics properties that follow solely from the assumption that a locally differentiable solution exists. The lack of any other assumptions on the optimization problem is what makes such properties basic, fundamental, or intrinsic. Although it is nearly universal, and indeed often sensible and justifiable, to make assumptions regarding the functional forms of the objective or constraint functions, or suppose that certain monotonicity or

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curvature properties hold on the said functions, these and other such ad hoc suppositions transcend those required to derive the fundamental comparative statics of a differentiable optimization problem. Accordingly, such assumptions do not yield intrinsic qualitative results and thus are not made in this work.

A similar development to that in static optimization took place in continuous time dynamic optimization, although it was initiated 43 years after the work of Samuelson (1947). In particular, Caputo (1990a) derived the basic comparative dynamics of an open-loop solution for a general class of variational calculus problems, while Caputo (1990b) did the same for an open-loop solution of a general class of optimal control problems. More than a decade later, Caputo (2003) essentially "closed the loop" by deriving the fundamental comparative dynamics of a closed-loop (or feedback) solution of the class of autonomous and exponentially discounted infinite horizon optimal control problems. In passing, note that Oniki's (1973) approach to comparative dynamics is complementary to that in the above papers, in that it produces results concerning the effects of parameter changes on an open-loop solution of a control problem, and is generally useful for problems with only one state variable.

Another similar development took place in the static (or atemporal) game theory, but this one was initiated 49 years after Samuelson's (1947) basic contribution. In this case, Caputo (1996) derived the intrinsic comparative statics of a general class of static games possessing locally differentiable Nash equilibria. It took only two more years before Caputo (1998) did the same for atemporal Stackelberg equilibria.

Given the preceding historical evolution, the goal of this paper is a wholly natural one, to wit, the derivation of the intrinsic comparative dynamics of locally differentiable feedback Nash equilibria for a ubiquitous class of differential games. The class of differential games contemplated are autonomous, exponentially discounted, have an infinite time horizon, and possess a feedback information structure for which its corresponding feedback Nash equilibria exist. This class of differential games is, arguably, the most widely employed and studied in economics. Inasmuch as the goal of the paper is a natural extension of the aforementioned literature, little motivation need be provided. Nonetheless, several remarks are worthwhile to make at this stage.

First, given that it is more difficult to derive a closed-form solution for differential games than for optimal control problems or static games, the results obtained herein go some way toward overcoming the fact that "it is difficult to obtain results from general differential games", (Reinganum (1982, p. 674)), seeing as the comparative dynamics results are obtained in a rather general setting. Second, the main result shows that all differential games of the said class possess refutable, and thus in principle, empirically testable comparative dynamics properties. Third, these properties are intrinsic to the studied class of differential games and thus should be tested or imposed in empirical work that uses the differential game theory as the basis for the development of an empirical model. And fourth, the intrinsic comparative dynamics take the form, in part, of generalized Slutsky-like expressions, i.e., one portion of the basic comparative dynamics take the form of linear combinations of partial derivatives of the feedback Nash equilibria with respect to the parameters and state variables of the game, while the other portion reflects the impact of the strategic nature of the differential game on its basic comparative dynamics.

As is often the case when one studies a general class of problems—and there is no exception in the present instance—certain structural elements of the class point to special cases which are wholly unanticipated, yet yield insight not possible in a typical tightly specified setting, and which have real utility in applied work. In particular, it is shown that if the instantaneous payoff function of every player is additively separable between that player's control variables and those of every other player, and if, in addition, all the transition functions are additively separable between the control variables of the different players, then the intrinsic comparative dynamics of differential games are of the same form as those in optimal control problems. Thus, even though a differential game is inherently strategic in nature, the above restrictions lead to intrinsic comparative dynamics in which the strategic element is absent. Moreover, the separability restrictions are common to many of the applied differential games appearing in the literature, as is documented in Section 3. Indeed, even the seemingly ubiquitous and workhorse class of linear–quadratic differential games satisfies the aforesaid separability conditions, thereby implying that many of the applied differential games solved in the literature possess intrinsic comparative dynamics in which the strategic component vanishes. Applications of the results to generalized capital accumulation and sticky-price games round out the contribution of the manuscript, and at the same time, impart some economic intuition to the results.

Finally, note that the following notational conventions are adopted: (i) all vectors are column vectors and are indicated in the boldface type, (ii) the derivative of a scalar-valued function with respect to a column vector is a row vector, and is indicated by a boldface subscript letter on the function, (iii) the derivative of a vector-valued function with respect to a vector is a Jacobian matrix, the number of rows of which equal the number of functions being differentiated and the number of columns of which equal the number of elements in the vector that the derivative is taken with respect to, and is indicated by a boldface subscript letter on the boldface function, and (iv) the symbol "†" indicates transposition.

2. Technical preliminaries

Consider the class of exponentially discounted and autonomous differential games consisting of a finite number $P \in \mathbf{Z}_{++}$ of players, indexed by $p \in \{1, 2, \ldots, P\}$, and played over a infinite time horizon, where $t \in [0, +\infty)$ is a fixed but arbitrary initial time, often referred to as a base time. The state of the differential game at each instant $\tau \in [t, +\infty)$ is given by the state vector $\mathbf{x}(\tau) \in X$, where $X \subseteq \mathbb{R}^N$ is an open set referred to as the state space of the game. The initial value of the state vector, denoted