Journal of Mathematical Economics 52 (2014) 87–97

Contents lists available at ScienceDirect

Journal of Mathematical Economics

journal homepage: www.elsevier.com/locate/jmateco

On the existence of mixed strategy Nash equilibria

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ARTICLE INFO

Article history:
Received 7 October 2013
Received in revised form
3 March 2014
Accepted 9 April 2014
Available online 23 April 2014

Keywords:
Discontinuous game
Diagonally transfer continuous game
Better-reply secure game
Mixed strategy equilibrium
Transfer lower semicontinuity

ABSTRACT

The focus of this paper is on developing verifiable sufficient conditions for the existence of a mixed strategy Nash equilibrium for both diagonally transfer continuous and better-reply secure games. First, we show that employing the concept of diagonal transfer continuity in place of better-reply security might be advantageous when the existence of a mixed strategy Nash equilibrium is concerned. Then, we study equilibrium existence in better-reply secure games possessing a payoff secure mixed extension. With the aid of an example, we show that such games need not have mixed strategy Nash equilibria. We provide geometric conditions for the mixed extension of a two-person game that is reciprocally upper semicontinuous and uniformly payoff secure to be better-reply secure.

1. Introduction

Nowadays, one of the main tools in the arsenal of economists concerned with equilibrium existence is Reny's (1999) theorem, according to which a compact Borel game has a mixed strategy Nash equilibrium if its mixed extension is better-reply secure.[1] In applications, better-reply security usually follows from two conditions: one related to reciprocal upper semicontinuity and the other to payoff security.

Establishing the payoff security of a game's mixed extension often constitutes a complicated problem. The concept of uniform payoff security, introduced by Monteiro and Page (2007), makes the problem considerably more tractable in games where it is applicable, including catalog games (Page and Monteiro, 2003) and voting models (Carbonell-Nicolau and Ok, 2007).[2] Verifying whether a game that is not upper semicontinuous-sum has a better-reply secure mixed extension is, as a rule, quite challenging. This paper's main focus is on studying the existence of a mixed strategy Nash equilibrium in normal form games where the sum of the payoff functions is not necessarily upper semicontinuous.

We begin with considering the games having a diagonally transfer continuous mixed extension, appealing to the analogy with the better-reply secure mixed extensions. Baye et al. (1993) showed that the existence of a pure strategy Nash equilibrium in diagonally transfer continuous games follows from a generalization of the Knaster–Kuratowski–Mazurkiewicz (KKM) lemma.[3] In Section 2 of this paper, the Ky Fan minimax inequality, in a slightly generalized form, is used to prove that every compact Borel game whose mixed extension is diagonally transfer continuous has a mixed strategy Nash equilibrium. The range of applications of this basic result is considerably broader than that of Glicksberg's (1952) equilibrium existence theorem — whose proof is based on the Kakutani–Fan–Glicksberg fixed point theorem. In particular, the mixed extension of a game is diagonally transfer continuous if the following two conventional assumptions hold: the extension is payoff secure and the game is upper semicontinuous-sum. Then, in Section 3, we extend the concept of uniform payoff security to diagonally transfer continuous games by introducing uniform diagonal security. In the upper semicontinuous-sum games, uniform payoff security implies uniform diagonal security. At the same time, if a compact Borel game is uniformly diagonally secure, it has a mixed

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[1] A number of results extending Reny's equilibrium existence theorem have been obtained recently (see, e.g., Barelli and Meneghel, 2013; Bich, 2009; Carmona, 2011; de Castro, 2011; McLennan et al., 2011 and Reny, 2013).

[2] Another approach to showing the payoff security of mixed extensions can be found in Duggan (2007), where hospitable strategies are used for studying equilibrium existence in voting models.

[3] The first part of this paper contains a number of results first presented in our 2012 working paper "On Uniform Conditions for the Existence of Mixed Strategy Equilibria".

http://dx.doi.org/10.1016/j.jmateco.2014.04.002
0304-4068/

strategy Nash equilibrium, which makes it possible to avoid having to study any additional properties of the game's mixed extension.

Example 1 is a slight modification of the Tullock rent-seeking game where it is additionally assumed that the favor the players vie for is granted to a third party with probability one-half if at least one player exerts no effort at all. Notwithstanding the fact that the game is not better-reply secure, it is not only diagonally transfer continuous, but also uniformly diagonally secure; that is, the game has a mixed strategy Nash equilibrium.

In Section 4, we adapt Simon's (1987) concept of weak domination on average to our setting by introducing weak uniform payoff security, a generalization of uniform payoff security. Using this concept, we construct a better-reply secure two-person game with a payoff secure mixed extension that has no mixed strategy equilibria (Example 2).

In Section 5, we study the existence of a mixed strategy equilibrium in reciprocally upper semicontinuous games that are uniformly payoff secure. In such games, the equilibrium existence problem becomes considerably more tractable if it is possible to transform the game into an upper semicontinuous-sum game with the aid of positive affine transformations,[4] which, in particular, implies that the game also has a reciprocally upper semicontinuous mixed extension. In Example 3, this technique is applied to a conventional two-candidate probabilistic spatial voting game. Theorems 5 and 6 give geometric sufficient conditions for games that are reciprocally upper semicontinuous and uniformly payoff secure to have a better-reply secure mixed extension.

The Appendix contains a number of auxiliary results, deferred proofs, and some comments regarding Theorem 5b of Dasgupta and Maskin (1986).

2. The model and some facts

We consider a game $G = (X_i, u_i)_{i\in I}$, where $I = \{1, \ldots, n\}$, each player i's pure strategy set X_i is a nonempty, compact subset of a metrizable topological vector space, and each payoff function u_i is a bounded Borel measurable function from the Cartesian product $X = \Pi_{i\in I}X_i$, equipped with the product topology, to $\mathbb{R}$. Under these conditions, $G = (X_i, u_i)_{i\in I}$ is called a compact Borel game. A game $G = (X_i, u_i)_{i\in I}$ is quasiconcave if each X_i is convex and $u_i(\cdot, x_{-i}) : X_i \to \mathbb{R}$ is quasiconcave for all $i \in I$ and all $x_{-i} \in X_{-i}$, where $X_{-i} = \Pi_{k\in I\setminus\{i\}}X_k$. In this paper, by a game we mean a compact Borel game.

The following definition of a payoff secure game is due to Reny (1999).

Definition 1. In $G = (X_i, u_i)_{i\in I}$, player i can secure a payoff of $\alpha \in \mathbb{R}$ at $x \in X$ if there exists $d_i \in X_i$ such that $u_i(d_i, x'_{-i}) \geq \alpha$ for all x'_{-i} in some open neighborhood of x_{-i}. The game G is payoff secure if for every $x \in X$ and every $\varepsilon > 0$, each player i can secure a payoff of $u_i(x) - \varepsilon$ at x.

Payoff security can be reformulated in terms of transfer lower semicontinuity, due to Tian (1992).

Definition 2. Let Z and Y be two topological spaces. A function $f : Z \times Y \to \mathbb{R}$ is λ-transfer lower semicontinuous in y if for every $(z, y) \in Z \times Y$, $f(z, y) > \lambda$ implies that there exists some point $z' \in Z$ and some neighborhood $\mathcal{N}_Y(y)$ of y in Y such that $f(z', w) > \lambda$ for all $w \in \mathcal{N}_Y(y)$. A function $f : Z\times Y \to \mathbb{R}$ is transfer lower semicontinuous in y if f is λ-transfer lower semicontinuous in y for every $\lambda \in \mathbb{R}$.

[4] Using a similar approach, Amir (2005) gives examples of Cournot oligopolies possessing the cardinal complementarity property where the other complementarity conditions are ineffective.

A game is payoff secure if and only if each player's payoff function is transfer lower semicontinuous in the other players' strategies (see Prokopovych, 2011, Lemma 1).

The graph of G is defined by $\mathrm{Gr}G = \{(x, u) \in X \times \mathbb{R}^n \mid u_i(x) = u_i \text{ for all } i \in N\}$, and the set of pure strategy Nash equilibria of G in X is denoted by E_G. For a subset B of a topological vector space X, we denote by cl$B$ the closure of B and by co$B$ the convex hull of B. In a metric space Y, we denote by $B_Y(y, r)$ the open ball centered at y and with radius $r > 0$.

Definition 3. A game $G = (X_i, u_i)_{i\in I}$ is better-reply secure if whenever $(x^*, u^*) \in \mathrm{clGr}G$ and $x^* \in X \setminus E_G$, some player i can secure a payoff strictly above u^*_i at x^*.

A useful fact is that a payoff secure game is better-reply secure iff it is also transfer reciprocally upper semicontinuous (see Bagh and Jofre, 2006 and Prokopovych, 2011, Lemma 2).

Definition 4. A game $G = (X_i, u_i)_{i\in I}$ is: (i) reciprocally upper semicontinuous if for any $(x, \alpha) \in \mathrm{clGr}G \setminus \mathrm{Gr}G$, there is a player i such that $u_i(x) > \alpha_i$; (ii) weakly reciprocally upper semicontinuous if whenever $(x, \alpha) \in \mathrm{clGr}G\setminus\mathrm{Gr}G$, there are a player i and $d_i \in X_i$ such that $u_i(d_i, x_{-i}) > \alpha_i$; (iii) transfer reciprocally upper semicontinuous if whenever $(x, \alpha) \in \mathrm{clGr}G \setminus \mathrm{Gr}G$ and x is not a Nash equilibrium, there are a player i and $d_i \in X_i$ such that $u_i(d_i, x_{-i}) > \alpha_i$.

It is clear that every weakly reciprocally upper semicontinuous game is transfer reciprocally upper semicontinuous.

Reny's (1999) equilibrium existence theorem states that every compact, quasiconcave, better-reply secure game has a Nash equilibrium in pure strategies.

Theorem 1 (Reny, 1999). *If $G = (X_i, u_i)_{i\in I}$ is compact, quasiconcave, and better-reply secure, then it possesses a pure strategy Nash equilibrium.*

Another approach to studying equilibrium existence in discontinuous games is based on the concept of diagonal transfer continuity, due to Baye et al. (1993).

For $G = (X_i, u_i)_{i\in I}$, define the following aggregator functions:

$$A_G : X \times X \to \mathbb{R} \quad \text{by } A_G(d, x) = \sum_{i\in I} u_i(d_i, x_{-i}),$$

where, as usual, the $-i$ subscript on x stands for "all players except i",

$$A^0_G : X \to \mathbb{R} \quad \text{by } A^0_G(x) = \sum_{i\in I} u_i(x),$$

and

$$F_G : X \times X \to \mathbb{R} \quad \text{by } F_G(d, x) = A_G(d, x) - A^0_G(x).$$

A strategy profile $x \in X$ is a Nash equilibrium of G iff $F_G(d, x) \leq 0$ for all $d \in X$.

Definition 5. A game $G = (X_i, u_i)_{i\in I}$ is diagonally transfer continuous if for every $x \in X\setminus E_G$, there exist some $d \in X$ and some neighborhood $\mathcal{N}_X(x)$ of x in X such that $F_G(d, z) > 0$ for all $z \in \mathcal{N}_X(x)$.

It is worth noticing that G is diagonally transfer continuous iff F_G is 0-transfer lower semicontinuous in x.

Every payoff secure game with an upper semicontinuous A^0_G is diagonally transfer continuous.

Lemma 1. *If, in a game $G = (X_i, u_i)_{i\in I}$, each $u_i : X \to \mathbb{R}$ is transfer lower semicontinuous in x_{-i} and the aggregator function $A^0_G : X \to \mathbb{R}$ is upper semicontinuous, then G is diagonally transfer continuous.*